A GENERALIZED PROCEDURE FOR DETERMINING TRANSIENT RESPONSE OF THIRD ORDER SYSTEMS

JOE P. HARRISON

A GENERALIZED PROCEDURE FOR DETERMINING TRANSIENT RESPONSE OF THIRD ORDER SYSTEMS

by

Joe P. Harrison

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ELECTRICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California

1 9 6 5

A GENERALIZED PROCEDURE FOR DETERMINING TRANSIENT RESPONSE OF THIRD ORDER SYSTEMS

by

Joe P. Harrison

This work is accepted as fulfilling

the thesis requirements for the degree of

MASTER OF SCIENCE

IN

ELECTRICAL ENGINEERING

from the

United States Naval Postgraduate School

ABSTRACT

For any system whose input-output relationship may be described by a ratio of polynomials in S (the complex number $\sigma \pm j\omega$), one of the measures of the system performance is its transient response to a step input. For systems of order greater than second the determination of the parameters of this transient response has always been a difficult process. In this thesis a new procedure is developed which allows the rapid and accurate determination of the transient parameters of any system up to and including third order with one zero. The procedure consists of developing the response equation in terms of angles at the singularities rather than the magnitudes. The resulting equation is utilized to develop expressions for times of extremals, magnitudes of extremals, and a form of rise time. A simple but accurate graphical method is demonstrated for solving the resulting transcendental equations. It is shown that the expression for times of extremals is a function of only two variables and may therefore be displayed on a two-dimensional plot. The times determined from this plot are utilized in computing the magnitudes of extremals. Various curves are presented to provide a means for general analysis.

TABLE OF CONTENTS

List of Illustrations

TABLE OF SYMBOLS AND ABBREVIATIONS

Symbol	Definition
S	The complex frequency ($S=\sigma \pm j\omega$).
σ	The real part of a complex number; neper frequency.
ω	The imaginary part of a complex number; radian frequency.
$\sigma_d \pm j\omega_d$	The S-plane coordinates of a pair of complex poles.
ω_n	The natural frequency ($\omega_n = \sqrt{\sigma_d^2 + \omega_d^2}$).
Z	The S-plane coordinate of a real zero.
P	The S-plane coordinate of a real pole.
γ	The angle measured clock wise from the negative real axis to the vector from the origin to the complex pole $\sigma_d + j\omega_d$.
θ_P	The angle measured counter clock wise from the real axis to the vector from a real pole to the complex pole $\sigma_d + j\omega_d$.
θ_Z	The angle measured counter clock wise from the real axis to the vector from a real zero to the complex pole $\sigma_d + j\omega_d$.
t	Real time.
T	Nondimensionalized time ($T = \omega_d t$).
t_{pn}	The time to the n^{th} point of zero slope of the transient response.
t_{cn}	The time to the n^{th} point at which the transient response is equal to the final value.
$X_{(t)}$	The time response.
M_{ptn}	Normalized magnitude of the transient response at time t_{pn} ($M_{ptn} = X_{(t_{pn})}/\text{final value}$).

1. Introduction.

In recent years the study of systems which have an input-output relationship (or a relationship between any two variables of a system) which is described by a differential equation has increased tremendously. Such relationships are found in the equations of motion of objects, in the equations for chemical reactions, in thermodynamic relations, in the equations for heat and fluid flow, in electrical network equations, in automatic control and servomechanism equations, and even in the equations governing the fields of industry and management. These relationships are written in either of two ways:

(a) Given that the variables of interest are X(t) and Y(t) which are expressed as functions of some third variable W(t) and its derivatives:

$$X(t) = a_n \, d^nW/dt^n + a_{n-1} \, d^{n-1}W/dt^{n-1} + \text{-------} + a_0 W$$

$$Y(t) = b_m \, d^mW/dt^m + b_{m-1} \, d^{m-1}W/dt^{m-1} + \text{-------} + b_0 W$$

(b) Given that the variables of interest are X(t) and Y(t) which are expressed implicitly in the equation:

$$b_m \, d^mX/dt^m + \text{-------} + b_0 X = a_n \, d^nY/dt^n + \text{-----} + a_0 Y$$

In either case the complex frequency ($s = \sigma \pm j\omega$) may be introduced and the relationship expressed as:

$$(1) \quad \frac{X(s)}{Y(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \text{-------} + a_0}{b_m s^m + b_{m-1} s^{m-1} + \text{-------} + b_0}$$

The usual case of interest is to allow Y(t) to be some input (either an initial condition or a specific function) and to investigate the response of X(t). A specific case which is often

mentioned in design specifications is the response of $X(t)$ when $Y(t)$ is a unit step input (i.e., a unit value input applied instantaneously at time zero). Depending on the order of the relationship , the value of M in the equation (1), the determination of the response may be straightforward or quite tedious and time consuming. For the pure second-order system ($m=2$ and $n=0$) the response has been extensively investigated, and general curves and equations are available in any elementary text on servomechanisms. For relationships with $m>2$ and/or $n>0$ the information is not nearly so plentiful. The approach taken by most authors is to present the equation in basic form, comment briefly on the effects of the higher order terms, and turn to methods of making these higher order terms negligible in order that the response may be analyzed in terms of the more familiar second-order relationships. Unfortunately, the higher order terms cannot always be made negligible, and the engineer's only recourse has been to work out each system in detail. No method has been available for a <u>general</u> analysis of the effects of the higher order terms. In this thesis a procedure is developed which enables the analyst or designer to solve in general an equation of one higher order in both numerator and denominator (i.e., order three in denominator and order one in numerator.)

Perhaps the simplest way to gain an appreciation of the problem is to examine the relationship on the S-plane. Equation (1) may be rewritten as follows:

$$1(a)\quad \frac{X(s)}{Y(s)} = k\,\frac{(s+Z_1)(s+Z_2)\cdots\cdots(s+Z_n)}{(s+P_1)(s+P_2)\cdots\cdots(s+P_m)}$$

Where the Zi and Pi may have either or both a real part and an imaginary part.

An example of such a system appears in Figure 1.

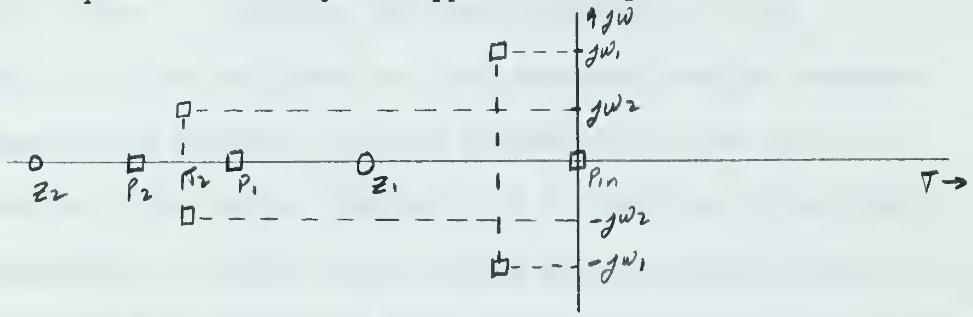


Figure 1: An example of S-plane diagramming

The system depicted would be written as:

$$X(s) = k\,\frac{(s+Z_1)(s+Z_2)}{s(s+P_1)(s+P_2)(s+\sigma_1+j\omega_1)(s+\sigma_1-j\omega_1)(s+\sigma_2+j\omega_2)(s+\sigma_2-j\omega_2)}$$

where the S term in the denominator represents Pin, the step input.

The usual nomenclature assigns the term poles to the P values (represented by □) and the term zeroes to the Z values (represented by o) with the collective name of singularities assigned to all roots of both the numerator and denominator polynomials. These singularities may occur anywhere in the S-plane and in any combination, but for stable physical systems the poles always occur in the left half plane ($\sigma<0$), and complex singularities always occur in conjugate pairs.

The second-order approximation approach is to consider the "dominant" pair of complex poles ($\sigma_1 \pm j\omega_1$ in the example) as the determining factor and force all other singularities far out in

the left half plane so that they may be considered negligible.

A generalized method of analysis for systems with one pair of complex poles and one real singularity has been developed by Patton and Abbott [1]. This method consists of using a scale factor to transform the real singularity to the $\sigma = -1$ position on the real axis and then determine various response parameters by entering a set of generalized curves with the location of the scaled complex poles in relation to the fixed singularity. A study of this method was the starting point in the search for a method of generalizing the equation for a system with a pair of complex poles, one real pole, and one real zero.

2. Attempts at Generalizing the Equation for a Third Order System with One Zero.

A clarification should be made of what is meant by generalizing the equation. If the stated problem contains more than two variables, it is desirable to combine the variables by means of some linear transformation so that a maximum of three transformed quantities remain as variables and all other variables are "buried" in the transformation.

This point is illustrated by means of two examples:

(1) The pure second-order response to a unit step input is expressed as:

$$X_{(s)} = \frac{k}{s(s^2 + 2\sigma_d s + \omega_n^2)}$$

Assume that a time (time of peak, rise time, settling time, etc.) is the transient parameter of interest. There are then three variables of interest: σ_d, ω_d, & t (k appears only as a multiplier). It is possible to display the information desired on a two-dimensional graph by displaying σ_d on the abscissa, t on the ordinate, and ω_d as a parameter for a family of curves; however,the problem may be simplified by noting that the time solution may be written as:

$$X_{(t)} = \frac{k}{\omega_n^2}\left[1 - \frac{\sqrt{\sigma_d^2 + \omega_d^2}}{\omega_d} e^{-\sigma_d t} \sin(\omega_d t + \tau)\right]$$

where $\tau = \text{Arctan}\left(\omega_d/\sigma_d\right)$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Now let $T = \omega_d t$, and $U = \sigma_d/\omega_d$:

$$X_{(t)} = {}^{k}\!/_{\omega_n^2}\left[1 - \sqrt{U^2+1}\, e^{-UT} \sin(T+\gamma)\right]$$

$$\gamma = \text{Arctan}\,(1/U)$$

This information could be displayed on a two-dimensional graph with U as abcissa and T as ordinate. The necessity for the family of curves has been precluded by "burying" the variable ω_d in the transformation $T = \omega_d t$.

(2) As a second example, consider the response of the pure third-order system to a unit step input:

$$X_{(s)} = \frac{k}{s(s+p)(s^2+2\sigma_d s+\omega_n^2)}$$

There are now four variables: σ_d, ω_d, t, & p. Obviously, four variables cannot be displayed on a two dimensional plane.

Writing the response in the time domain:

$$X_{(t)} = \frac{k}{\omega_n^2 p}\left[1 - \left(\frac{\omega_d^2+\sigma_d^2}{\omega_d^2+(p-\sigma_d)^2}\right)e^{-pt} - \frac{\sqrt{\sigma_d^2+\omega_d^2}\; p\, e^{-pt}}{\omega_d\sqrt{\omega_d^2+(p-\sigma_d)^2}}\sin(\omega_d t+\gamma-\theta_p)\right]$$

$$\gamma = \text{Arctan}\,(\omega_d/\sigma_d) \;;\; \theta_p = \text{Arctan}\,(\omega_d/p-\sigma_d)$$

Now let $T = \omega_d t$: $U = \sigma_d/\omega_d$ and $V = p/\omega_d$:

$$X_{(T/\omega_d)} = \frac{k}{\omega_n^2 p}\left\{1 - \frac{1+U^2}{1+(V-U)^2}e^{-VT} - \frac{\sqrt{1+U^2}}{\sqrt{1+(V-U)^2}}Ve^{-UT}\sin(T+\gamma-\theta_p)\right\}$$

$$\gamma = \text{Arctan}\,(1/U) \;;\; \theta_p = \text{Arctan}\left(\frac{1}{V-U}\right)$$

This information may be displayed on a two-dimensional graph by plotting U on the abcissa, T on the ordinate and V as a parameter for a family curves. Again, the fourth variable has been "buried" in the transformation $T = \omega_d t$.

It may be noted that the transformation used in the above examples was not the only possibility. In either examples, another

possibility would have been $T=\tau t$; and in the case of the third order system, another possibility would have been $T=\rho t$. The latter is the transformation used by Patton and Abbott [1] who showed that this transformation is equivalent to placing the pole at minus one on a W-plane defined by $W=s/p$. These authors then displayed the response parameters of rise time, time of peak overshoot, magnitude of peak overshoot, and settling time by providing curves on the W-plane representing constant values of these parameters. The parameter values for a particular system may be determined by locating the conjugate roots on the W-plane and reading the value of the curve passing through these points.

The response of a third-order system with one zero to a unit step input is expressed as:

$$X(s) = \frac{K(s+Z)}{s(s+P)(s^2+2\sigma_d s+\omega_n^2)}$$

The number of variables has now been increased to five: ω_d, σ_d, t, P, & Z. This number can be reduced to four by means of a transformation similar to the type just described, but a maximum of three may be displayed on a two-dimensional curve. In order to reduce the problem a transformation must be found which "buries" two of the variables. One approach is made by noting that the pure third order system was generalized by making a transformation which placed the real pole at the minus one position on a W-plane. It appears that if a transformation could be found which would fix the two real singularities at two predetermined points on some

W-plane where W is a linear function of S then the only variables would be the angle and distance of the conjugate poles from the origin of this W plane. In order for this transformation to be useful there must exist a unique relationship between the fictitious time associated with the W-plane and real time associated with the S-plane, and likewise such a relationship must exist between fictitious magnitude and real magnitude.

Although many minor attempts were made to determine a transformation of this type, only two seemed to have any merit:

(1) The bilinear transformation.

It is well known that by means of the bilinear transformation any three points on the S-plane can be mapped into any three points on the W-plane. If it is desired to map the points P, Z, and origin into the points -1, -2, and the origin, this may be achieved by the transformation:

$$\frac{(W+1)(-2)}{W[-2-(-1)]} = \frac{(S-P)Z}{(Z-P)S}$$

which reduces to:

$$W = \frac{2(P-Z)S}{(Z-P)S+PZ} \quad ; \quad S = \frac{PZW}{W(2P-Z)+2(P-Z)}$$

(2) Transformation by scaling and translation.

Another means of placing the pole at the minus one point and the zero at the minus two point is by compressing (or expanding) the scale until the distance from the pole to the minus one point is equal to the

distance from the zero to the minus two point and then translating through this distance. This may be accomplished by the transformation:

$$W = \frac{1}{(P-Z)}\left[S + (Z-2P)\right]$$

$$S = (P-Z)\left[W + \frac{(2P-Z)}{(P-Z)}\right]$$

Although either of these transformations can be shown to be suitable for certain specific cases no unique relationships could be determined which would apply for the general case.

No further attempts were made at determining a transformation. Although no proof is given, it will be shown in the next section that it is unlikely that a suitable transformation exists.

One of the major difficulties experienced by Patton and Abbott even in view of the generalizing transformation, was the solution of a transcendental equation of the form:

$$k_1 e^{-a\omega t} = \sin(\omega t + k_2)$$

In computing values for the curves in their thesis they were forced to use iterative methods for the solution. While studying their work a graphical method was discovered for solving this form of equation, and this discovery led to the idea for a means of generalizing the third order system.

3. Development of the General Equation.

The response of the third order system with one zero is expressed in the S domain as:

$$X_{(s)} = \frac{k(S+Z)}{S(S+P)(S^2+2\sigma_d S+\omega_n^2)} = \frac{k(S+Z)}{S(S+P)(S+\sigma_d+j\omega_d)(S+\sigma_d-j\omega_d)}$$

and is expressed in the time domain as:

$$X(t) = \frac{kZ}{(\sigma_d^2+\omega_d^2)P} + \frac{k(P-Z)}{P(P^2-2P\sigma_d+\sigma_d^2+\omega_d^2)}e^{-Pt}$$

$$-\frac{ke^{-\sigma_d t}}{(\sigma_d^2+\omega_d^2)}\left\{\frac{[\sigma_d(P-Z)+(\sigma_d-Z)(\sigma_d-P)+\omega_d^2]}{(\sigma_d-P)^2+\omega_d^2}\cos\omega_d t\right.$$

$$\left.+\frac{[\sigma_d[(\sigma_d-Z)(\sigma_d-P)+\omega_d^2]-\omega_d^2(P-Z)}{[(\sigma_d-P)^2+\omega_d^2]\,\omega_d}\sin\omega_d t\right\}$$

The time domain equation is derived by simple residue methods and algebraic manipulation. The various values are depicted in Figure 2.

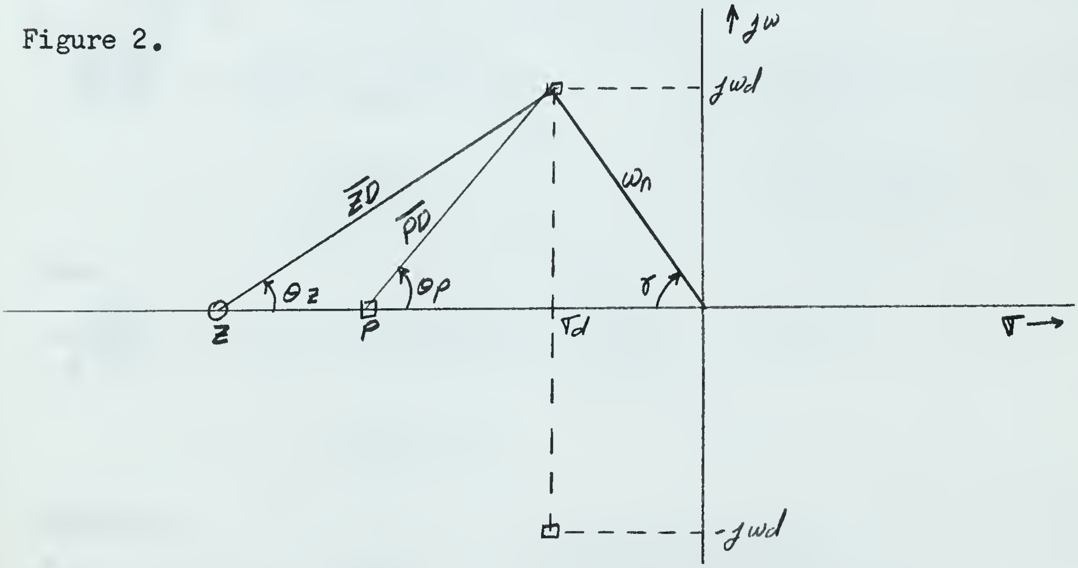


Figure 2: S-plane plot of $\dfrac{S+Z}{S(S+P)(S^2+2\sigma_d S+\sigma_d^2+\omega_d^2)}$

The expression for X(t) may be simplified by rewriting it in

terms of the angles θp, θz, and γ. The following relationships are written by inspection from Fig. 2:

$$\sigma_d^2 + \omega_d^2 = \omega_n^2$$
$$\overline{PD}^2 = (P-\sigma_d)^2 + \omega_d^2$$
$$\overline{ZD}^2 = (Z-\sigma_d)^2 + \omega_d^2$$
$$\sin\gamma = \omega_d/\omega_n ; \cos\gamma = \sigma_d/\omega_n$$
$$\sin\theta p = \omega_d/\overline{PD} ; \cos\theta p = P-\sigma_d/\overline{PD}$$
$$\sin\theta z = \omega_d/\overline{ZD} ; \cos\theta z = Z-\sigma_d/\overline{PD}$$

By trigonometric identities:

$$\sin(\theta z - \theta p) = \frac{\omega_d (P-Z)}{\overline{PD}\,\overline{ZD}}$$

$$\cos(\theta z - \theta p) = \frac{(Z-\sigma_d)(P-\sigma_d) + \omega_d^2}{\overline{ZD}\,\overline{PD}}$$

From these relationships:

$$X(t) = \frac{KZ}{\omega_n^2 P}\left\{1 + \frac{\omega_n \overline{ZD}}{Z\sin\gamma\,\overline{PD}}\sin(\theta z - \theta p)e^{-Pt} - \frac{P\overline{ZD}\,e^{-\sigma_d t}}{Z\,\overline{PD}\sin\gamma}\sin(\omega_d t + \theta z - \theta p + \gamma)\right\}$$

By further use of the relationships above:

$$\frac{\omega_n \overline{ZD}}{Z\overline{PD}} = \frac{\omega_n\ \omega_d/\sin\theta z}{\omega_d(\cot\theta z + \cot\gamma)\frac{\omega_d}{\sin\theta p}} = \frac{\omega_n \sin\theta p}{\omega_n \sin\theta z \sin\gamma(\cot\theta z + \cot\gamma)}$$

$$= \frac{\sin\theta p}{\cos\theta z \sin\gamma + \sin\theta z \cos\gamma} = \frac{\sin\theta p}{\sin(\theta z + \gamma)}$$

And:

$$\frac{P\overline{ZD}}{Z\overline{PD}} = \frac{\omega_d(\cot\theta p + \cot\gamma)\ \omega_d/\sin\theta z}{\omega_d(\cot\theta z + \cot\gamma)\ \omega_d/\sin\theta p} \cdot \frac{\sin\gamma}{\sin\gamma}$$

$$= \frac{\sin(\theta p + \gamma)}{\sin(\theta z + \gamma)}$$

Therefore:

$$\text{Eq'n. I}: X(t) = \frac{KZ}{\omega_n^2 P}\left\{1 + \frac{\sin\theta p \sin(\theta z - \theta p)}{\sin\gamma \sin(\theta z + \gamma)} e^{-[\cot\theta p + \cot\gamma]\omega_d t} - \frac{\sin(\theta p + \gamma)}{\sin\gamma \sin(\theta z + \gamma)} e^{-(\cot\gamma)\omega_d t} \sin(\omega_d t + \theta z - \theta p + \gamma)\right\}$$

Eq'n. I has a number of important properties which are noted here:

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

[illegible]

(1) For stable systems the multiplying constant $\left\{\frac{kz}{\omega_n^2 p}\right\}$ is the final value determined by:

$$\text{Final Value} = \lim_{s \to 0} sX(s)$$

(2) If the transformation $T = \omega_n t$ is made the equation is completely non-dimensionalized; i.e., the equation is a function of angles only.

(3) The equation may be normalized to unity by multiplying by $k_2 = \frac{\omega_n^2 p}{kz}$; i.e. the final value has been normalized to unity. Using these properties:

$$\text{Eq'n. Ia:} \quad X_n(T/\omega_n) = 1 + \frac{\sin\theta p \sin(\theta z - \theta p)}{\sin\gamma \sin(\theta z + \gamma)} e^{-(\cot\theta p + \cot\gamma)T} - \frac{\sin(\theta p + \gamma)}{\sin\gamma \sin(\theta z + \gamma)} e^{-(\cot\gamma)T} \sin(T + \theta z - \theta p + \gamma)$$

Eq'n Ia is valid for $\theta z \neq 180^\circ - \gamma$ and $\theta p < 180^\circ - \gamma$; i.e. for the zero not at the origin and the pole in the left half plane (it is assumed that $\gamma \leq \pi/2$). When $\theta p \geq 180^\circ - \gamma$ the system is unstable and the final value theorem does not hold. When $\theta z = 180^\circ - \gamma$ the zero is at the origin and the final value is zero. These special cases are treated in detail in later sections.

Eq'n Ia is valid for any order system up to and including third order with one zero. Any combination may be achieved by allowing either or both of the angles θp and θz to go to zero which is equivalent to locating the singularity at infinity.

This equation may now be manipulated to determine various parameters of the time response.

4. Development of the Equations for the Parameters of the Time Response.

For a given system the slope of the response curve at time zero can be quickly computed along with the final value of the response. Since it is also known that any oscillatory motion is of the damped sinusoid form it is necessary to know only three other parameters to be able to sketch quite accurately the complete transient response. These parameters are:

(1) tc-time at which the response crosses the final value.

(2) tp-time of extremals (points where slope of time response is zero.)

(3) Mpt-normalized magnitude of the response at the extremals.

$$[M_{pt} = X_n(t_p)]$$

(a) Development of equation for the time of extremals (tp):

In order to determine this value the expression for X(t) is differentiated and equated to zero:

$$\frac{d}{dt}[X(t)] = \frac{KZ}{\omega_n^2 p}\left\{\left(\frac{\sin\theta_p \sin(\theta_z-\theta_p)}{\sin\gamma \sin(\theta_z+\gamma)} e^{-pt}\right)(-p) - \left(\frac{\sin(\theta_p+\gamma)}{\sin\gamma \sin(\theta_z+\gamma)} e^{-\omega_d t/\tan\gamma}\right)\left[\omega_d \cos(\omega_d t+\theta_z-\theta_p+\gamma) - \frac{\omega_d}{\tan\gamma}\sin(\omega_d t+\theta_z-\theta_p+\gamma)\right]\right\}$$

equating to zero:

$$-p\sin\theta_p \sin(\theta_z-\theta_p)e^{-pt_p} = \sin(\theta_p+\gamma)e^{-\omega_d t_p/\tan\gamma}\,\omega_n[\sin\gamma\cos(\omega_d t_p+\theta_z-\theta_p+\gamma)-\cos\gamma\sin(\omega_d t_p+\theta_z-\theta_p+\gamma)]$$

$$-\omega_n \sin\gamma[\cot\theta_p+\cot\gamma]\sin\theta_p\sin(\theta_z-\theta_p)e^{-(p-\sigma_d)t_p} = \sin(\theta_p+\gamma)\,\omega_n\sin(-\omega_d t_p-\theta_z+\theta_p)$$

$$-\cancel{\omega_n}\,\cancel{\sin(\theta_p+\gamma)}\sin(\theta_z-\theta_p)e^{-(\cot\theta_p)\omega_d t_p} = -\cancel{\omega_n}\,\cancel{\sin(\theta_p+\gamma)}\sin(\omega_d t_p+\theta_z-\theta_p)$$

so:

$$\text{Eq'n. II:}\quad \sin(\theta_z-\theta_p)e^{-\omega_d t_p/\tan\theta_p} = \sin(\omega_d t_p+\theta_z-\theta_p)$$

There is no "closed" method for solving this equation for tp, but a simple graphical method exists and will be explained presently. Note that the times of the extremals are independent of the angle σ.

(b) Developments of the equation for normalized magnitude of extremals (Mpt): To determine this value the equation for X(t) is entered with the results of part (a):

$$M_{pt} = X(t_p)\Big/\left\{\frac{k_z}{\omega_n^2 p}\right\} = \left\{1 + \frac{\sin\theta_p \sin(\theta_z-\theta_p)}{\sin\sigma \sin(\theta_z+\sigma)} e^{-(p-\sigma)t_p} e^{-\sigma t_p} - \frac{\sin(\theta_p+\sigma)}{\sin\sigma \sin(\theta_z+\sigma)} e^{-\sigma t_p} \sin(\omega_d t_p+\theta_z-\theta_p+\sigma)\right\}$$

Substituting from Eq'n II for the quantity in braces:

$$\text{Eq'n III}: M_{pt} = \left\{1 + \frac{\sin(\theta_p+\sigma)}{\sin\sigma \sin(\theta_z+\sigma)} e^{-\omega_d t_p/\tan\sigma}\left[\frac{\sin\theta_p}{\sin(\theta_p+\sigma)} \sin(\omega_d t_p+\theta_z-\theta_p) - \sin(\omega_d t_p+\theta_z-\theta_p+\sigma)\right]\right\}$$

Values for solution of this equation are determined graphically at the same time Eq'n II is solved.

(c) Development of the equation for the times when the time response crosses the final value (tc):

This time occurs when the last two terms of Eq'n I are equated to zero:

$$\frac{\sin\theta_p \sin(\theta_z-\theta_p)}{\sin\sigma \sin(\theta_z+\sigma)} e^{-p t_c} = \frac{\sin(\theta_p+\sigma)}{\sin\sigma \sin(\theta_z+\sigma)} e^{-\sigma t_c} \sin(\omega_d t_c+\theta_z-\theta_p+\sigma)$$

$$\text{Eq'n IV} \quad \frac{\sin\theta_p \sin(\theta_z-\theta_p)}{\sin(\theta_p+\sigma)} e^{-\omega_d t_c/\tan\theta_p} = \sin(\omega_d t_c+\theta_z-\theta_p+\sigma)$$

Again tc may be determined graphically on the same plot used to solve Eq'ns II and III.

The notation of the equations is simplified by making the transformations as before: $\begin{cases} T = \omega_d t \\ k_2 = \omega_n^2 p / k_z \end{cases}$

The transformed equations then become:

$$\text{Eq'n. IIa:} \quad \sin(\theta_z - \theta_p)\, e^{-T_p/\tan\theta_p} = \sin(T_p + \theta_z - \theta_p)$$

$$\text{Eq'n. IIIa:} \quad X_n\left(\frac{T_p}{\omega d}\right) = M_{pt} = \left\{1 + \frac{\sin(\theta_p + \gamma)}{\sin\gamma \sin(\theta_z + \gamma)}\, e^{-T_p/\tan\gamma} \left[\frac{\sin\theta_p}{\sin(\theta_p + \gamma)} \sin(T_p + \theta_z - \theta_p) - \sin(T_p + \theta_z - \theta_p + \gamma)\right]\right\}$$

$$\text{Eq'n. IVa:} \quad \frac{\sin\theta_p \sin(\theta_z - \theta_p)}{\sin(\theta_p + \gamma)}\, e^{-T_c/\tan\theta_p} = \sin(T_c + \theta_z - \theta_p + \gamma)$$

5. Graphical Solution of the Equations.

Equations IIa and IVa may be solved graphically to an accuracy of approximately one percent by following the step by step procedure outlined below. Equation IIIa may be solved directly after the solution of Equation IIa for Tp.

(a) Graphical solution is performed on semi-log paper with values of T plotted on the linear axis and values of the functions of T plotted on the logarithmic axis.

(b) Compute or measure the angles θp, θz, and γ on the S-plane diagram of the problem to be solved.

(c) Compute the following values:

$$\text{Sin}\ (\theta z-\theta p)$$
$$\text{Sin}\ \theta p/\text{Sin}\ (\theta p+\gamma)$$

(d) If $\text{Sin}(\theta z-\theta p) > 0$ Construct the function Sin T.

If $\text{Sin}(\theta z-\theta p) < 0$ Construct the function Sin(-T).

(Note: This function is easily constructed by plotting the values of Sin T for $T = 10^\circ, 20^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ and appropriate multiples. For $T < 10^\circ$ the function is approximately a straight line.)

(e) Construct the function $\text{Sin}(\theta z-\theta p)\, e^{-T/\tan \theta p}$.

(Note: On the semi-log paper this function is a straight line with intercept $\text{Sin}(\theta z-\theta p)$ and slope of $-\frac{1}{\tan \theta p}$.)

(f) Locate the point on the straight line where the directed horizontal distance to the sine function is equal to $(\theta z-\theta p)$. The value of T at this point is Tp.

(g) Construct the function $\frac{\sin\theta_p \sin(\theta_z-\theta_p)}{\sin(\theta_p+\tau)} e^{-T/\tan\theta_p}$

(Note: If $\frac{\sin\theta_p \sin(\theta_z-\theta_p)}{\sin(\theta_p+\tau)}$ is of the opposite sign to $\sin(\theta_z-\theta_p)$ it will be necessary to make a second plot on semi-log paper following the rules laid down in (d) above.)

(h) Locate the point on the second straight line where the directed horizontal distance to the sine function is equal to $(\theta_z-\theta_p+\tau)$. The value of T at this point is Tc.

(i) Compute the value of Mpt by use of Equation IIa and the value of Tp.

(j) Use the inverse transformations:

$$t = T/\omega_d$$

$$X(t_p) = \frac{1}{k_2} M_{pt} = \frac{k_z}{\omega_n^2 p} M_{pt}$$

to obtain true parameter values.

(k) Repeat steps (f), (h), and (i) for as many values as necessary to obtain a complete description of the transient response.

The crucial step in the graphical solution just outlined is the determination of the sign placed on the values of the logarithmic axis. Since the construction is performed on semi-log paper it is impossible to show both positive and negative values on the same plot; consequently, only alternate half-cycles of SinT may be plotted. If $\theta_z < \theta_p$ the value of $\sin(\theta_z-\theta_p)$ is negative and the logarithmic axis must be labelled with negative values; therefore, the negative portion of SinT must be plotted. Likewise, if $\theta_z > \theta_p$

the value of $\sin(\theta z-\theta p)$ is positive and positive portions of SinT must be plotted. Similar rules apply to the solution of Equation IVa.

The procedure is illustrated by means of the following example:

Given a system for which:

$$\theta_p = 80^\circ \qquad \omega_d = 1$$
$$\theta_z = 40^\circ \qquad \frac{kz}{\omega_n^2 p} = 1$$
$$\gamma = 84^\circ$$

The computation proceeds as follows (see Fig. 3):

(1) Construct the function SinT on semi-log paper. (Note: $\theta z-\theta p < 0$; therefore, plot the negative half-cycles).

(2) Construct the line $\sin(\theta z-\theta p)\, e^{-T/\tan\theta p}$

(Note: $\theta z-\theta p < 0$; therefore, $\sin(\theta z-\theta p) < 0$)

(3) The equation to be solved is:

$$-.643\, e^{-T/5.66} = \sin(T-40^\circ)$$

Locate the points where the horizontal distance between the straight line and the sine function plot is 40° (straight line point to the <u>right</u> of the sine function point). The T axis value of the points on the straight line are the values Tp:

$$T_{p1} = 4.1 \text{ radians}$$
$$T_{p2} = 6.82 \text{ radians}$$
$$T_{p3} = 10.3 \text{ radians}$$

Note that these values do not depend on the value of γ but only on the relative location of the complex poles and the real singularities.

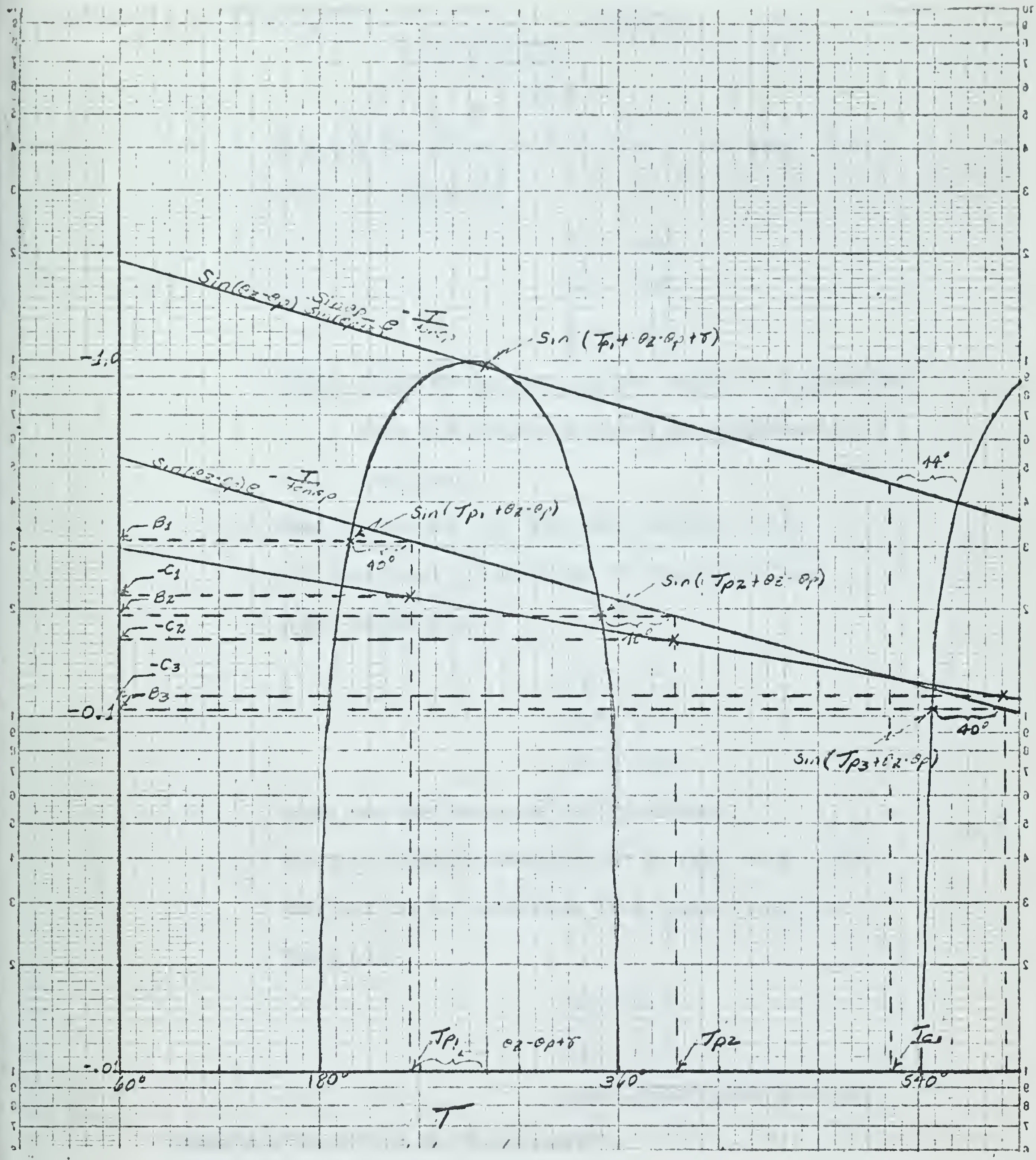


Fig.3: Computational Diagram for the

System: $\theta z = 40^\circ$; $\theta p = 80^\circ$; $\delta = 44^\circ$

(4) Construct the line;

$$\frac{\sin(\theta p+\tau)}{\sin\tau\,\sin(\theta z+\tau)}\,e^{-T/\tan\tau}$$

$$= .335\,e^{-T/9.5}$$

Read off the values of this line at the times Tpn:

(Call the values Cn.)

$C_1 = .213$

$C_2 = .165$

$C_3 = .113$

(Note: These values are positive since the logarithmic axis was assumed positive for construction of the line.)

(5) Read the values from the sine function at the correspondence points found in step (3): (Call these values Bn.)

$B_1 = -\ .31$

$B_2 = -\ .19$

$B_3 = -\ .103$

These are the values of Sin(Tpn+θz−θp).

(6) Add the distance $(\theta z-\theta p+\tau)=44^\circ$ to each value of Tpn and read the corresponding sine value: (call the value An.)

$A_1 = -\ 1.0$

$A_2 = -\ 1.0$

$A_3 = \ \mathrm{Sin}433^\circ = \sin 73^\circ = .955$

These are the values Sin(Tpn+θz−θp+τ).

(7) Compute the normalized magnitudes of the extremals:

$$M_{pt_n} = 1 + \frac{\sin(\theta_p + \tau)}{\sin\tau \sin(\theta_z + \tau)} e^{-T_{pn}/\tan\tau} \left[\frac{\sin\theta_p}{\sin(\theta_p+\tau)} \sin(T_{pn} + \theta_z - \theta_p) - \sin(T_{pn} + \theta_z - \theta_p + \tau) \right]$$

$$= 1 + C_n \left[3.58 B_n - A_n \right]$$

$M_{pt_1} = .9766$
$M_{pt_2} = .730$
$M_{pt_3} = 1.0715$

(8) Construct the line

$$\frac{\sin\theta_p}{\sin(\theta_p+\tau)} \sin(\theta_z - \theta_p) e^{-T/\tan\theta_p}$$
$$= -2.3\, e^{-T/5.66}$$

(9) The equation to be solved is:

$$-2.3\, e^{-T/5.66} = \sin\,(T + 44^\circ)$$

Proceed as in step (3) to find these points of correspondence. The value of T at these points corresponds to the times when the response crosses the final value (Tc):

$T_{c_1} = 9.1$ radians
$T_{c_2} = 11.6$ radians

This completes the values necessary.

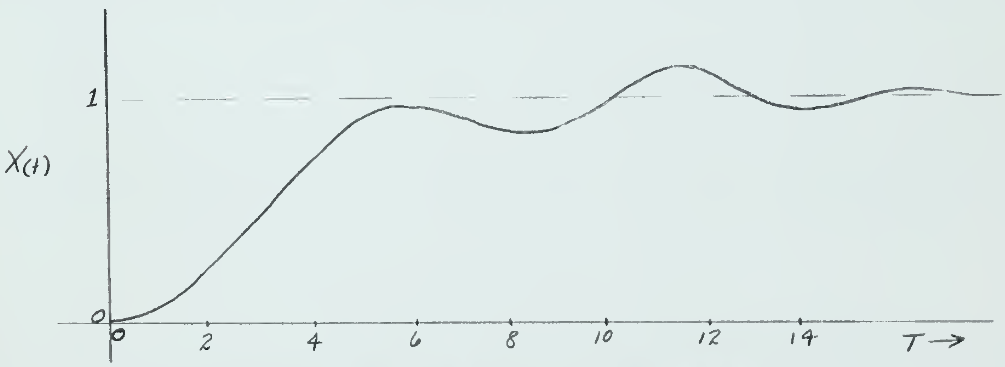


Figure 4: Step response for $\theta_z = 40^\circ$; $\theta_p = 80^\circ$; $\tau = 84^\circ$; $\omega_d = 1$

In order to demonstrate the accuracy of the method, this system was simulated on the CDC1604 digital computer using a

method of solution completely independent of the results obtained here. The computer response is shown in Figure 5 (Note: ω_d on the computer simulation was 2.0). For purposes of comparison, the computed values and the computer response values are tabulated:

Parameter	Computed	Computer Response	Difference
Tp_1	4.1	4.3	6.8%
Tp_2	6.82	6.9	2.5%
Tp_3	10.3	10.3	.96%
Mpt_1	.9766	.968	1.2%
Mpt_2	.730	.728	.55%
Mpt_3	1.0715	1.076	.51%
Tc_1	9.1	9.1	1.1%
Tc_2	11.6	11.5	1.75%

(Note: Computer response values were interpolated from the computer readout tabulation given in 0.1 second steps)

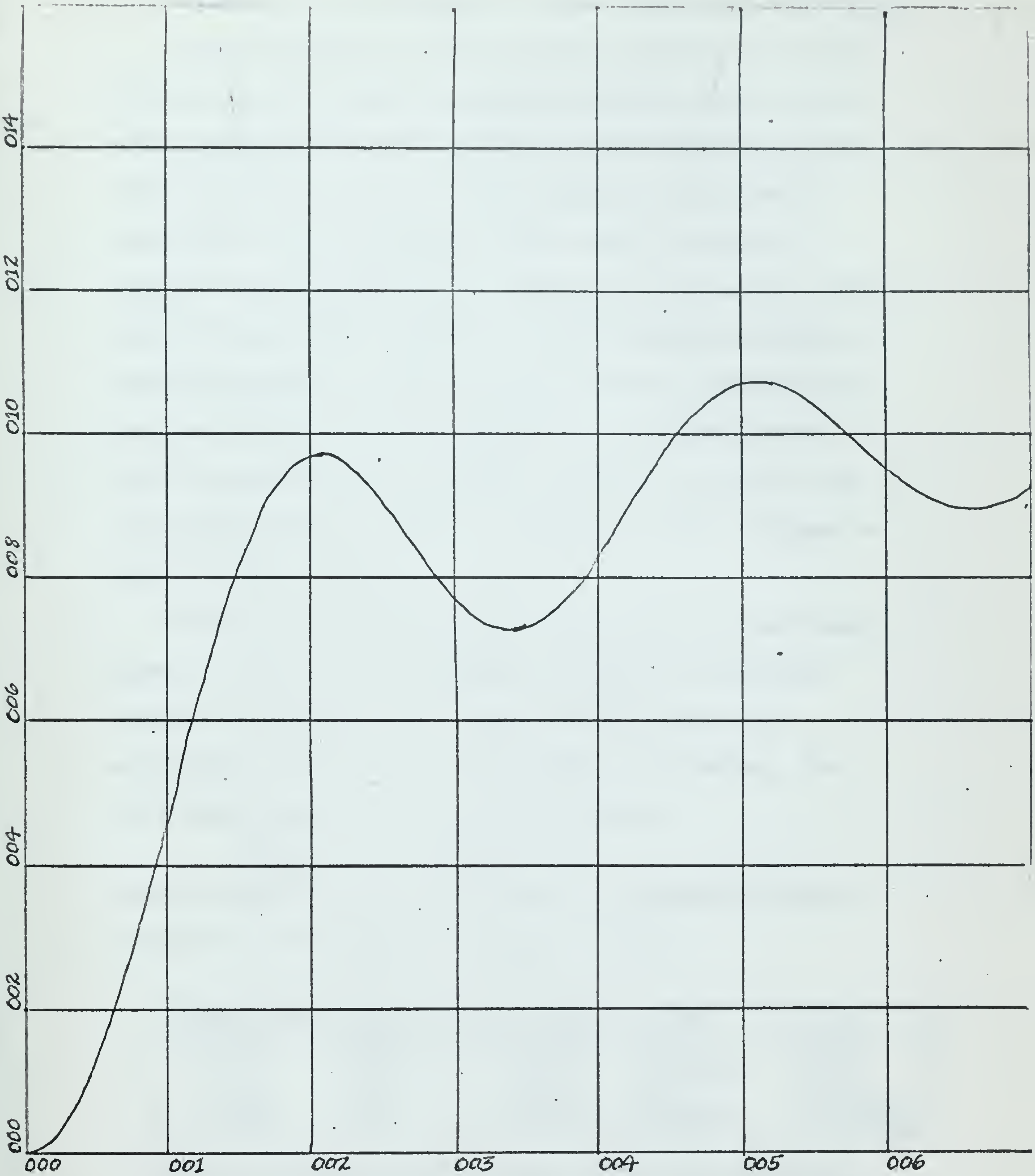


Fig.5: Computer Response for the

System: $\theta z = 40^\circ; \theta p = 80^\circ; \gamma = 44^\circ$.

6. Adaptation of the Equations to Other Singularity Combinations.

It has been stated that the equations developed are general in that they are correct for pure second-order systems, second-order systems with one zero, pure third order systems, and third order systems with one zero. This property is developed in more detail in this and later sections. Also, although an important value of the work just described is the method of solving the transcendental equations to obtain various parameters of the time response, an equally important value is that the means have been provided to develop curves and relationships which allow the analyst and/or designer to observe the effects of the real singularities at a glance. This point is also developed in detail in later sections.

To make equations I through IV applicable to the particular system of concern, it is necessary to modify the multiplying constant $\left\{kz/\omega_n^2 p\right\}$. It is more correct to define the multiplying constant as the "Final Value" of the system. The final value theorem for stable systems states:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$$

The final values of the various systems are tabulated in Table 1 along with the values of the angles θz and θp.

System Order	Zero	Final Value	θz	θp
Two	None	k/ω_n^2	Zero	Zero
Two	One	kz/ω_n^2	Variable	Zero
Three	None	$k/\omega_n^2 p$	Zero	Variable
Three	One	$kz/\omega_n^2 p$	Variable	Variable

Table 1: Modifying Relationships for Equations I through IV.

As an example equations I through IV (modified) are listed for order two with no zeros (pure second order):

$$\text{Eq'n. I (mod): } X(t) = \frac{k}{\omega_n^2}\left\{1 - \frac{1}{\sin\tau} e^{-(\cot\tau)\omega_d t} \sin(\omega_d t + \tau)\right\}$$

$$\text{Eq'n. II (mod): } 0 = \sin \omega_d t_p \Rightarrow \omega_d t_{p_n} = n\pi$$

$$\text{Eq'n III (mod): } M_{pt_n} = 1 - e^{-(\cot\tau)n\pi} \cos n\pi$$

$$\text{Eq'n IV (mod): } 0 = \sin(\omega_d t_c + \tau) \Rightarrow \omega_d t_{c_n} = n\pi - \tau$$

These equations are well known for the second order system although the form may be different.

7. Analysis of the second order system with one zero.

Following Table 1 of the last section:

$$\text{Eq'n. I (mod)}: \; X(t) = \frac{kz}{\omega_n^2}\left\{1 - \frac{1}{\sin(\theta_z+\gamma)} e^{-(\cot\gamma)\omega_d t} \sin(\omega_d t + \theta_z + \gamma)\right\}$$

$$\text{Eq'n. II (mod)}: \; 0 = \sin(\omega_d t_p + \theta_z) \Rightarrow \omega_d t_{p_n} = n\pi - \theta_z$$

$$\text{Eq'n. III (mod)}: \; M_{pt} = \left\{1 - \frac{e^{-(\cot\gamma)\omega_d t_{p_n}}}{\sin(\theta_z+\gamma)} \sin(\omega_d t_{p_n} + \theta_z + \gamma)\right\}$$

$$= \left\{1 - \frac{\cos n\pi \sin\gamma}{\sin(\theta_z+\gamma)} e^{-\cot\gamma\,(n\pi-\theta_z)}\right\}$$

$$\text{Eq'n. IV (mod)}: \; 0 = \sin(\omega_d t_c + \theta_z + \gamma) \Longrightarrow \omega_d t_{c_n} = n\pi - \theta_z - \gamma$$

which are the governing equations of the second order system with one zero. It may be noted that each of the time response parameters may be expressed as the pure second order parameter with a correction factor. In the case of the times of extremals and times when response crosses the final value, the correction is a linear one; i.e., the time is decreased by an amount θ_z/ω_d. The correction factor for the magnitude of extremals is a multiplicative nonlinear factor. The magnitude of extremals may be written:

X(tp) = Final Value (1 + % overshoot)

and: % overshoot (second order) = $-\cos(n\pi)\, e^{-(\cot\gamma)n\pi}$

and: % overshoot (second order with one zero) = $-\cos n\pi \left(\frac{\sin\gamma}{\sin(\theta_z+\gamma)}\right) e^{-(\cot\gamma)[n\pi-\theta_z]}$

$$= \left[\text{\% overshoot (second order)}\right] \frac{\sin\gamma}{\sin(\theta_z+\gamma)} e^{(\cot\gamma)\theta_z}$$

the correction factor = $\left[\frac{\sin\gamma}{\sin(\theta_z+\gamma)} e^{\theta_z \cot\gamma}\right]$

Figure 6 is a plot of Mpt of a second order system with one zero versus $\cos\gamma$ for various values of θz. The curve for $\theta z=0$ is a plot of the percent overshoot of a pure second order system. The system fails when $\theta z = \pi - \gamma$; i.e. when the zero is at the origin of the S-plane. At this point the final value is zero but

computed percent overshoot becomes infinite resulting in an undefined value. It is necessary to return to magnitudes to interpret this special case:

$$X_{(t_p)} = \frac{kZ}{\omega_n^2} - \frac{kZ}{\omega_n^2} \cos n\pi \frac{\sin\tau}{\sin(\theta_Z+\tau)} e^{-\cot\tau(n\pi-\theta_Z)}$$

$$= \frac{kZ}{\omega_n^2} - k\cos n\pi \frac{Z}{\omega_n^2} \frac{\omega_d}{\omega_n} \frac{e^{-\cot\tau(n\pi-\theta_Z)}}{\frac{\omega_d \tau_d}{\omega_n \bar{Z}_D} + \frac{Z-\tau_d}{\bar{Z}_D} \frac{\omega_d}{\omega_n}}$$

$$= \frac{kZ}{\omega_n^2} - k\cos n\pi \frac{Z}{\omega_n^2} \frac{\bar{Z}_D}{Z} e^{-\cot\tau(n\pi-\theta_Z)}$$

letting $\theta_Z = \pi - \tau$; $\bar{Z}_D = \omega_n$; $Z = 0$

$$X_{(t_p)}\Big|_{Z=0} = -\frac{k}{\omega_n} \cos n\pi \; e^{-\cot\tau[(n-1)\pi+\tau]}$$

for the curves of Figure 6, n=1 .

$$\therefore X_{(t_p)}\Big|_{Z=0} = \frac{k}{\omega_n} e^{-\tau\cot\tau}$$

The equations obtained here are quite simple to use and no further analysis of the system will be undertaken in this thesis.

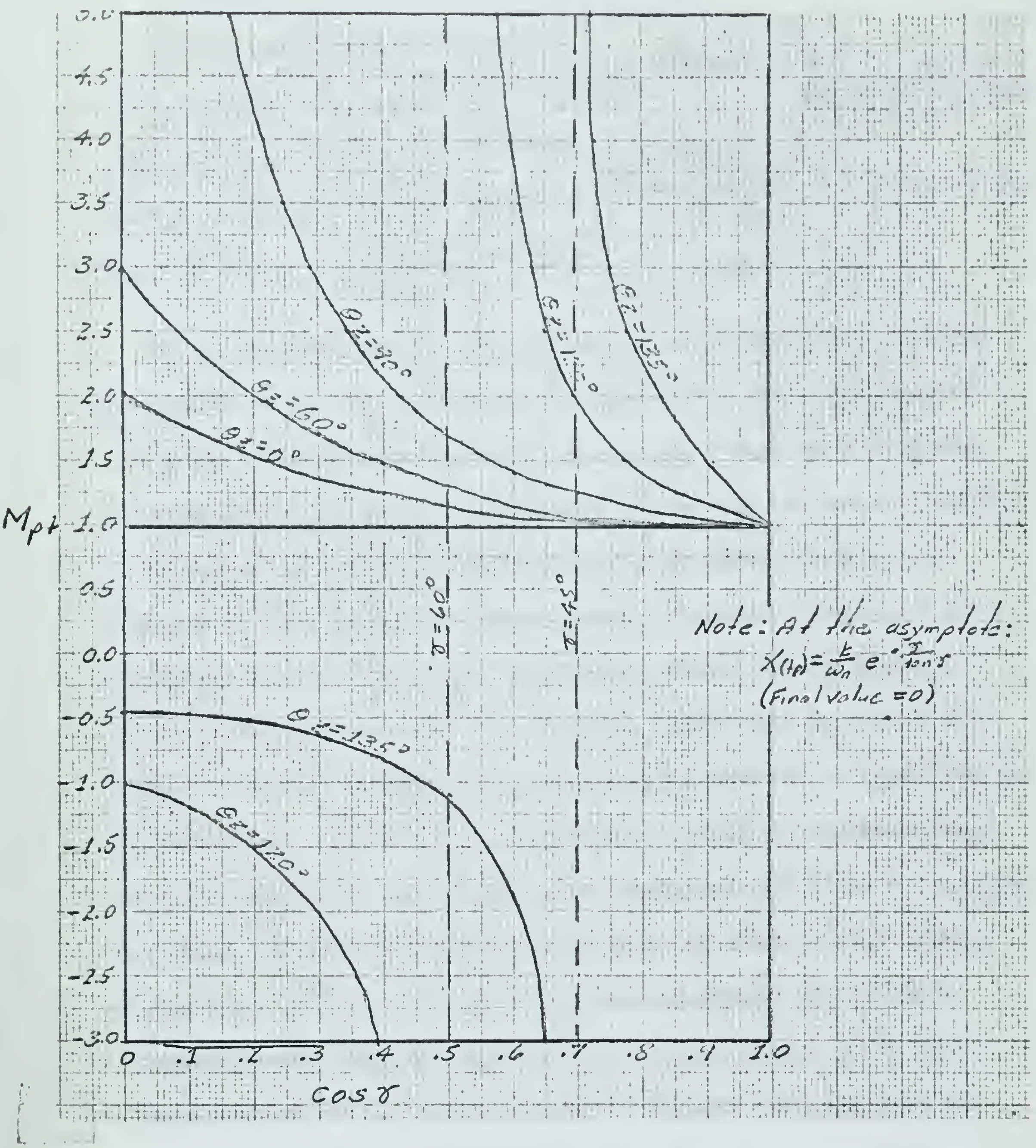


Fig.6: Magnitude of First Extremal of a

Second Order System with One Zero

(Final value normalized to unity).

8. Analysis of the pure third order system.

Following Table 1 of Section 6:

Eq'n. I (mod): $$X_{(t)} = \frac{k}{\omega_n^2 p}\left\{1 - \left(\frac{\sin\theta p}{\sin\gamma}\right)^2 e^{-[\cot\theta p + \cot\gamma]\omega_d t} - \frac{\sin(\theta p+\gamma)}{\sin^2\gamma} e^{-(\cot\gamma)\omega_d t} \sin(\omega_d t + \gamma - \theta p)\right\}$$

Eq'n. II (mod): $$-\sin\theta p \, e^{-\omega_d t_p/\tan\theta p} = \sin(\omega_d t_p - \theta p)$$

Eq'n. III (mod): $$M_{pt} = \left\{1 + \frac{\sin(\theta p+\gamma)}{\sin^2\gamma} e^{-\omega_d t_p/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\omega_d t_p - \theta p) - \sin(\omega_d t_p + \gamma - \theta p)\right]\right\}$$

Eq'n. IV (mod): $$-\frac{\sin^2\theta p}{\sin(\theta p+\gamma)} e^{-\omega_d t_c/\tan\theta p} = \sin(\omega_d t_c + \gamma - \theta_c)$$

It is obvious that no simple correction factor exists for the parameters as it did in the case of the second order system with one zero. The methods developed in section 5 have been utilized to develop curves for the parameters of the time response. Figure 7 is a plot of T_p versus θp (Note that T_p is independent of the angle γ). The curve has been extended to provide times for the occurence of the first three extremals (points of zero slope). It is interesting to note that the time correction is practically linear to $\theta p = 70^\circ$ and may be approximately expressed as $T_p = n\pi + \theta p$ up to that point. It will be shown presently that no overshoot can exist for $\theta p > 90^\circ$. For purposes of comparison the times of extremals are shown for a second order system with one zero (dashed lines). Figure 8(a) is a plot of the normalized magnitude of the first extremal of the response as a function of the cosine of γ for various values of θp (final value=1). Figures 8(b) and 8(c) are the same type of plot, but for the normalized magnitudes of the second and third extremals. The curve entitled $\theta p = 0$ is the familiar second order curve. An important point should be noted here.

For a second order system the first overshoot is always the greatest; but this is not necessarily the case for a third order system. For $\theta p > 76^{\circ}$, and dependent on the value of τ, the first "overshoot" may actually be less than the final value (in such a case the term "overshoot" is misleading); for $\theta p > 45^{\circ}$, and dependent on the value of τ, the second "overshoot" may be greater than the first. These effects are readily apparant upon examination of Figures 8(a) and 8(c). An interesting point is the case of $\theta p > 90^{\circ}$; on the semi-log paper used for computational purposes the line defined by the following equation:

$$y = \sin \theta p \, e^{-T/\tan \theta p}$$

is a line of positive slope and can never intersect the curve defined by the equation:

$$y = -\sin T$$

therefore, in accordance with the methods developed in Section 5, no time of overshoot can occur. This implies that for $\theta p > 90^{\circ}$, regardless of the value of τ, the time response is monotonically increasing and never exceeds the final value. Figure 9 is a plot of tc for the third order system (the rise time, tc, is defined to be the first time when the response is exactly equal to the final value). For values of $\theta p > 76^{\circ}$ it is possible, for certain values of τ, that the response does not cross the final value until after the first peak. It should be remarked here that the curves presented in Figures 7-9 are intended to serve as a means of general analysis and not necessarily as computational aids. The analyst or designer interested in exact values should perform the calculations as outlined in Section 5.

An interesting point in the study of the third order system is the value $\theta p=76^{o}$. This value appears to be the optimum setting to minimize fluctuations around the final value regardless of the value of τ; the response always reaches the final value on first rise and the first undershoot is minimum for all τ.

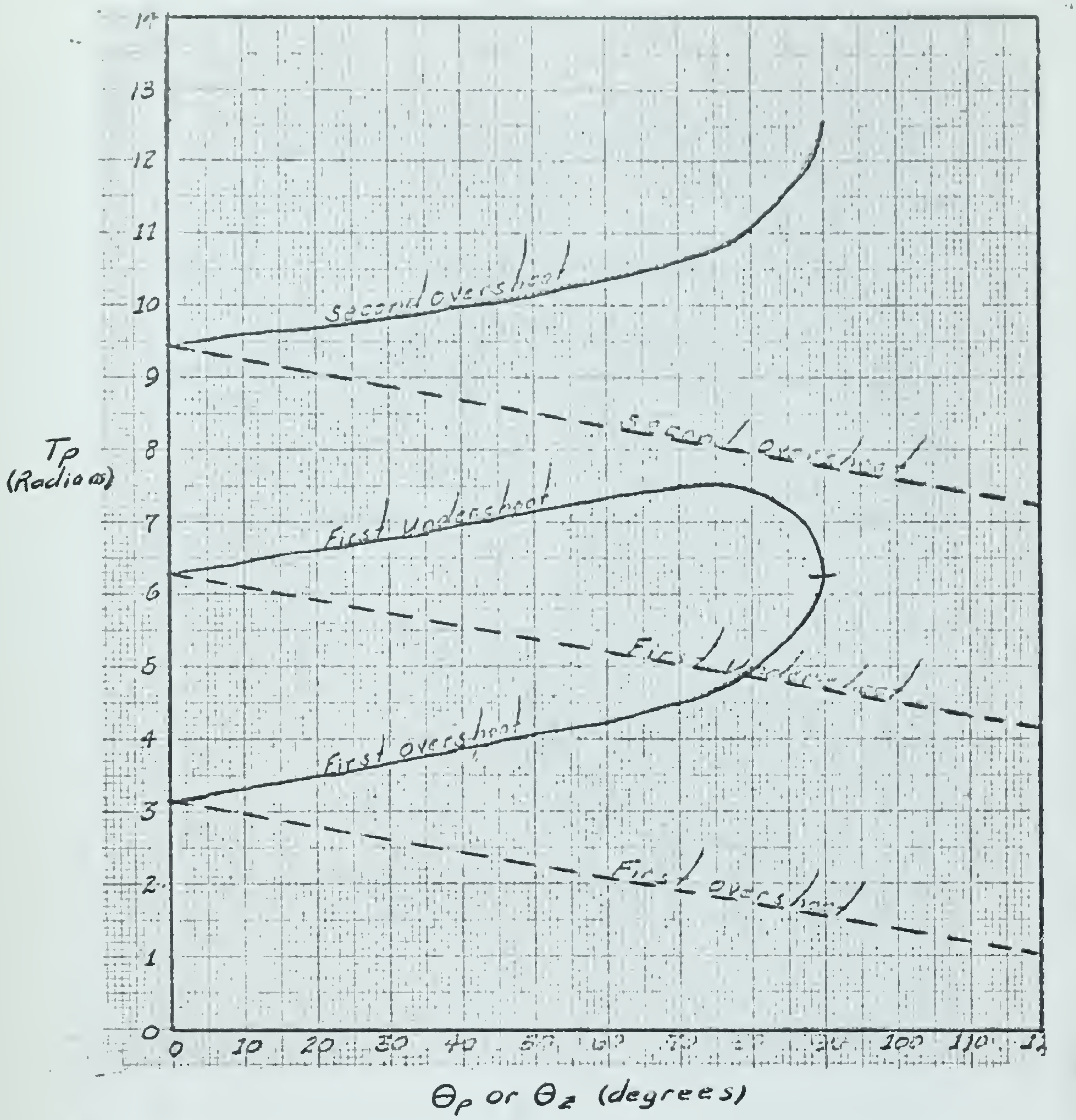
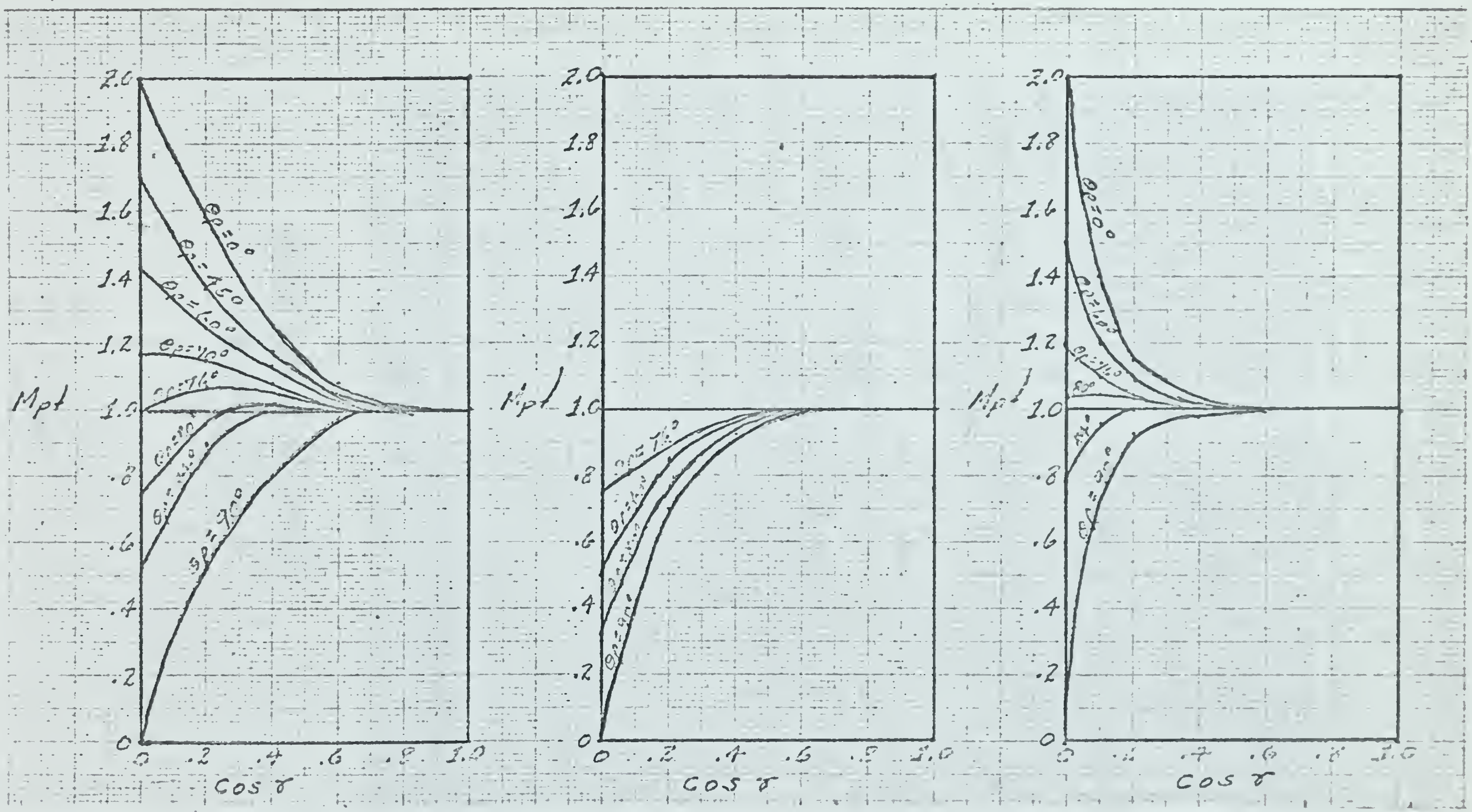


Fig.7: Times of Extremals of Step
Response:
(a)Solid- Third Order System.
(b)Dashed- Second Order System.



(a)First Extremal.

(b)Second Extremal.

(c)Third Extremal.

Fig.8: Magnitudes of First Three Extremals of the Step Response of a Third Order System (Final value normalized to unity).

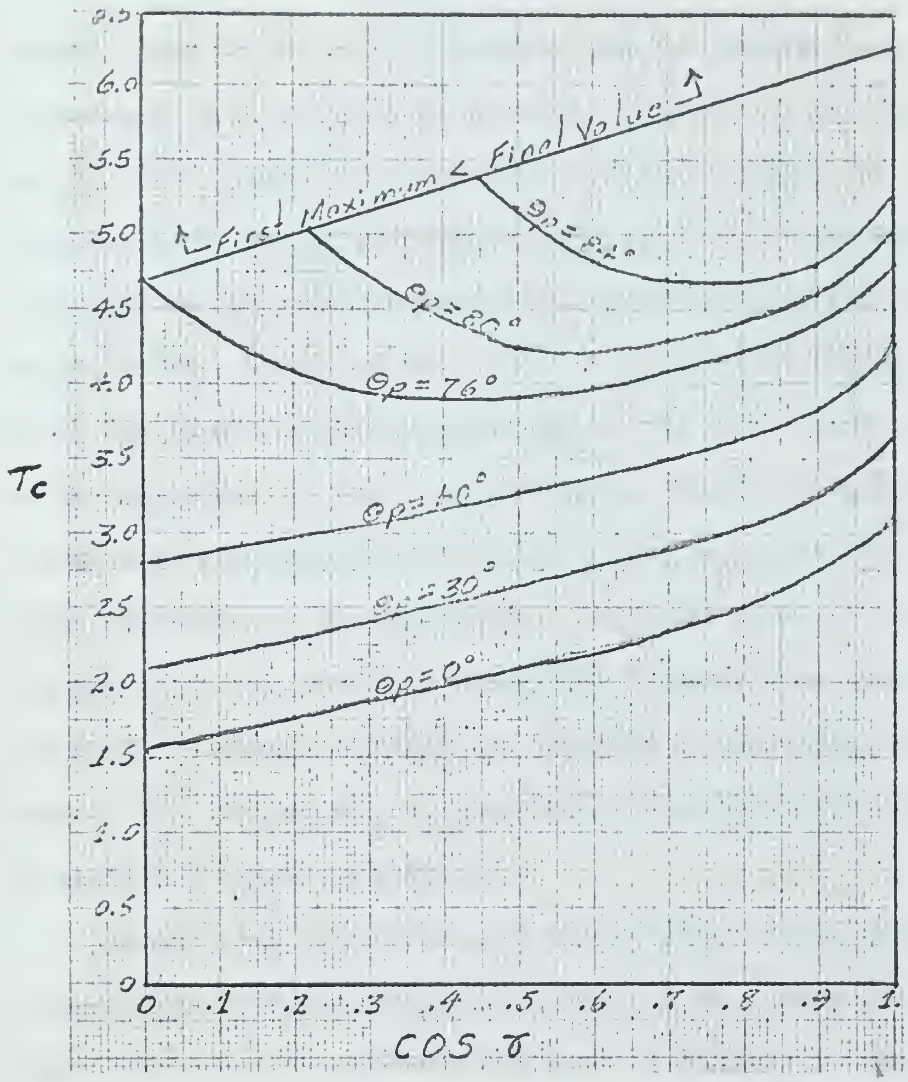


Fig.9: Rise Time of Third

Order Systems.

9. Analysis of the Third Order System with One Zero.

The character of the response of the third order system with one zero is best analyzed by plotting the values of Tp as shown in section 5, this information may be depicted on a two-dimensional plot since Tp is dependent only on the values of θz and θp. This plot may be constructed rather quickly and easily with Equation II by fixing θp (which fixes the slope of the straight line plot on the semi-log paper) and determining Tp for various values of θz. Curves of this form are displayed in Figures 10(a), 10(b) and 10(c) for Tp corresponding to the first three extremals of the response. If the values of θp and θz are known these curves may be entered and the value of Tp for each of the first three extremals may be determined. Once this value is known, the magnitudes of the peaks may be quickly computed from Equation III. Again, it is emphasized that the accuracy on the curves is only about $\pm$ 5%. If accuracy is required, the computational methods of section 5 should be utilized.

A great deal of knowledge of the behavior of the system response can be determined from a study of the curves of Figure 10. This information is summarized in part in Figure 11. The coordinate plane used in Figure 10 is depicted again in Figure 11, but the Tp curves are deleted and the plane is divided into regions as follows:

Region I $\begin{cases} \theta p < 90^\circ \\ \theta z > \theta p \end{cases}$

Region II $\begin{cases} \theta p < 90^\circ \\ \theta z < \theta p \end{cases}$

Region III $\begin{cases} \theta p > 90^\circ \\ \theta z < \theta p \end{cases}$

Region IV $\begin{cases} \theta p > 90^\circ \\ \theta z > \theta p \end{cases}$

These regions are formed by the line $\theta p = 90^\circ$, which is the angle at which the real pole moves "inside" the complex poles, and the line $\theta p = \theta z$ which is the "second order" line. The important factors of these divided zones are as follows:

(1) $\theta p < 90^\circ$ implies a negative slope to the line which is used for computation as described in Section 5. $\theta p > 90^\circ$ implies a positive slope.

(2) $\theta p > \theta z$ implies that $\mathrm{Sin}(\theta z - \theta p) < 0$, and $\mathrm{Sin}(\theta z - \theta p)$ is the y-axis intercept of the line described above. $\theta p < \theta z$ implies that $\mathrm{Sin}(\theta z - \theta p) > 0$. If $\mathrm{Sin}(\theta z - \theta p) < 0$ then the negative half cycles are plotted on the computational diagram, i.e. the values of the Sin T for $T = \pi$ to 2π, 3π to 4π, 5π to 6π, etc. If $\mathrm{Sin}(\theta z - \theta p) > 0$ then the positive half cycles are plotted i.e. the values of Sin T for $T = 0$ to π, 2π to 3π, etc. (See Section 5).

(3) If the slope of the computational line $\mathrm{Sin}(\theta z - \theta p)\, e^{-T/\tan \theta p}$ is negative then the line must intersect all the half cycles of Sin T and an infinite number of extremal points exist. If the slope is positive then it intersects the function Sin T

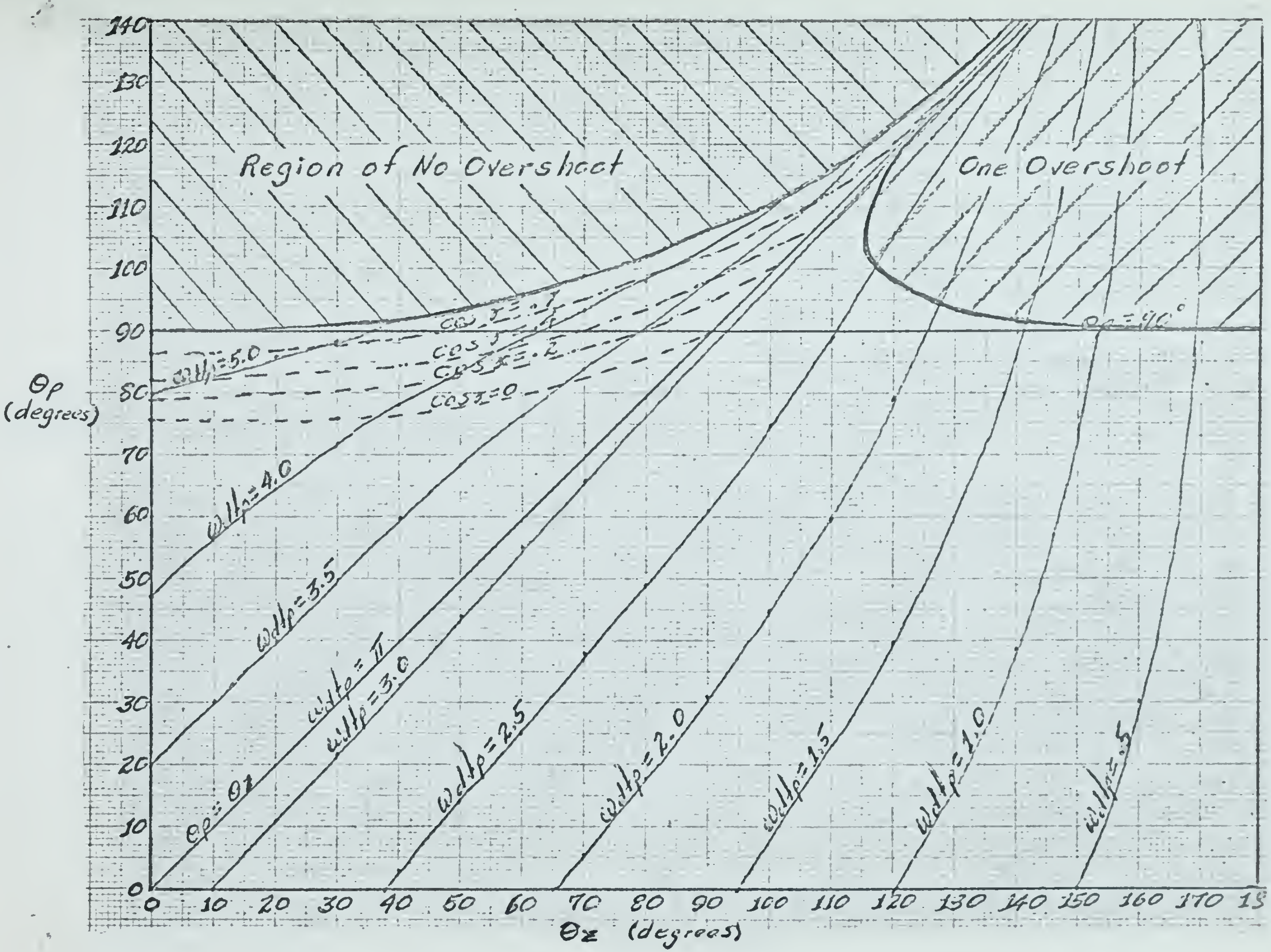


Fig.10a: Time of First Response Extremal of a Third Order System with One Zero.

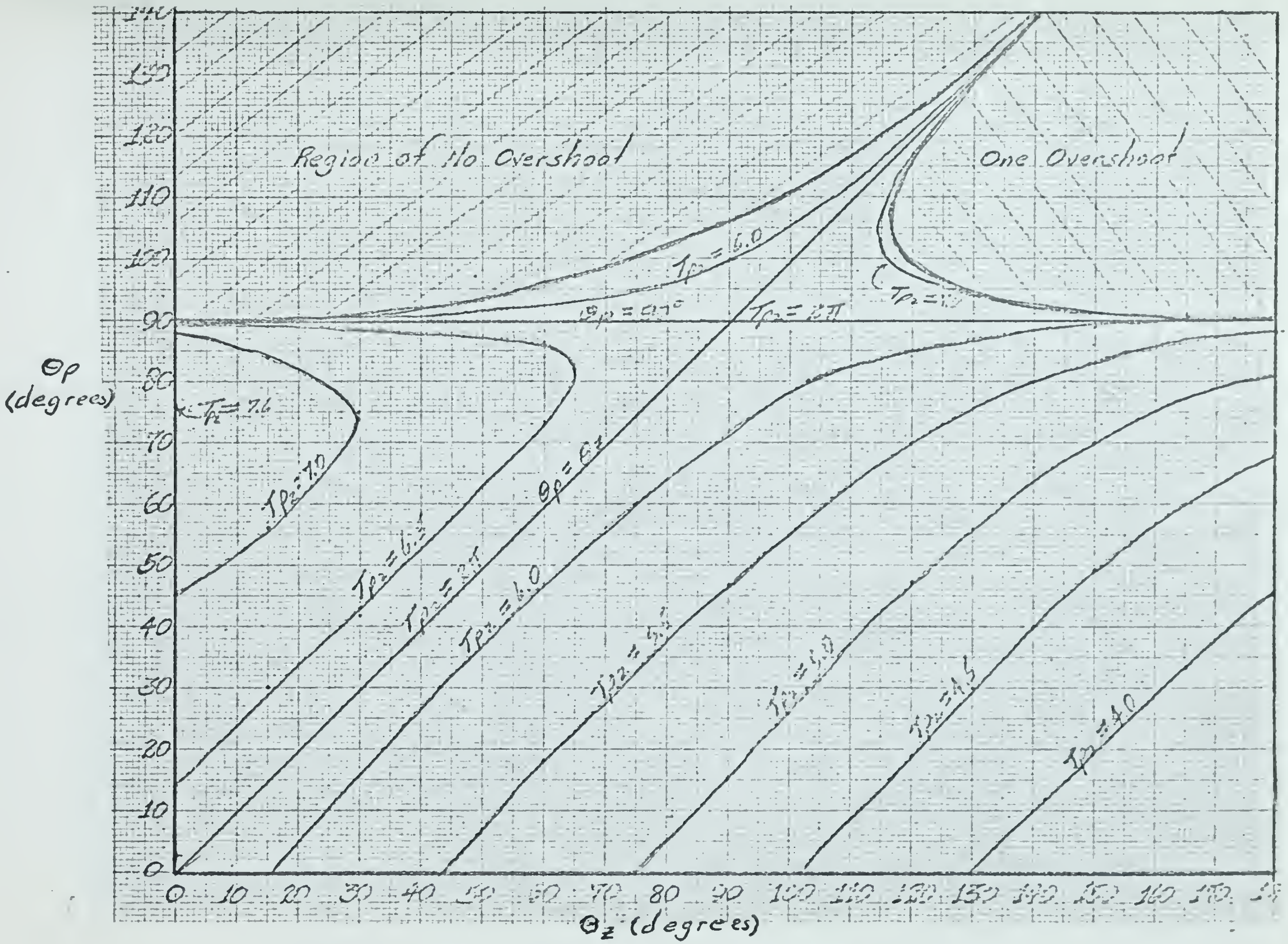


Fig. 10b: Time of Second Response Extremal of a Third Order System with One Zero.

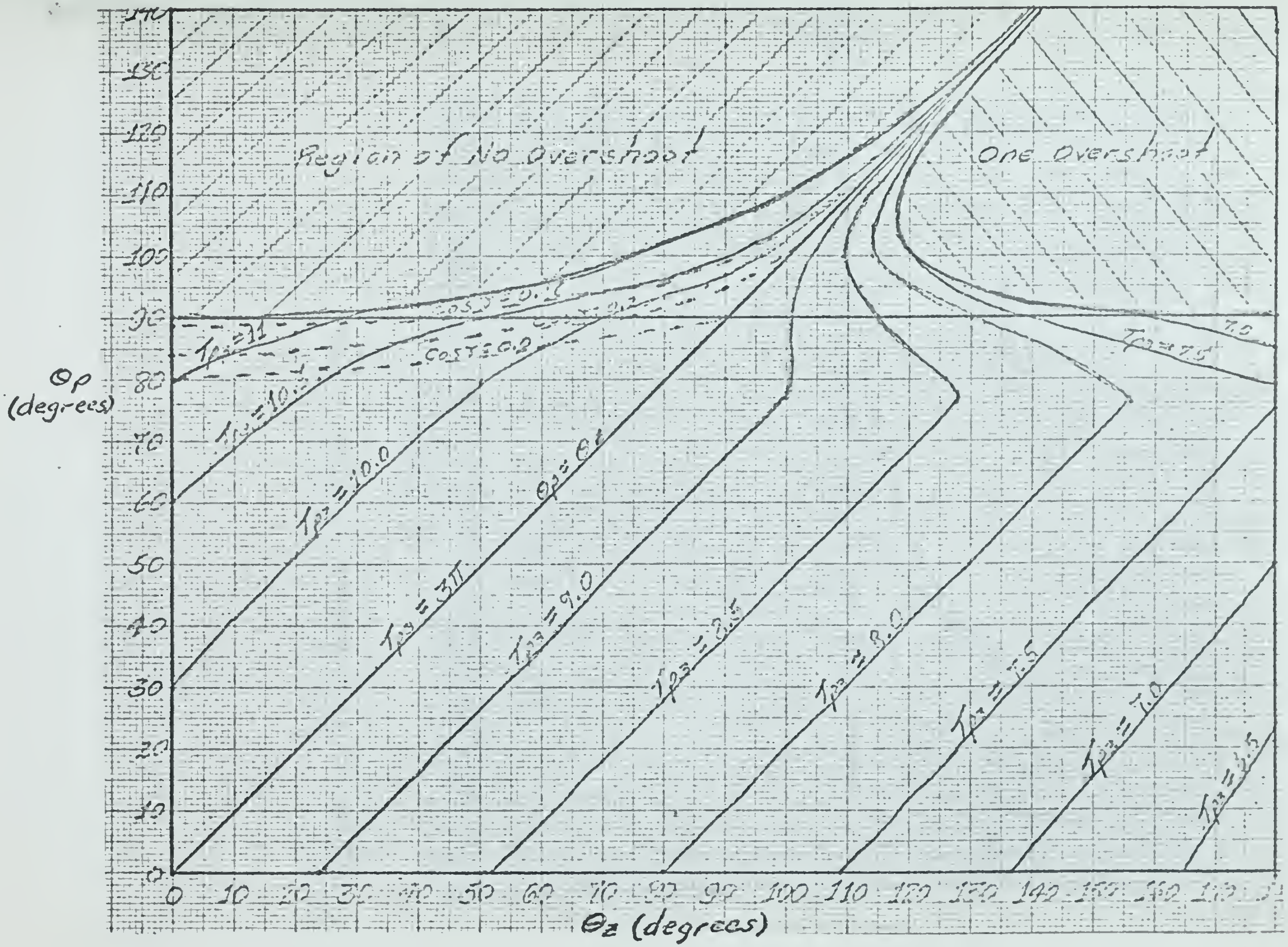


Fig. 10c: Time of Third Response Extremal of a Third Order System with One Zero.

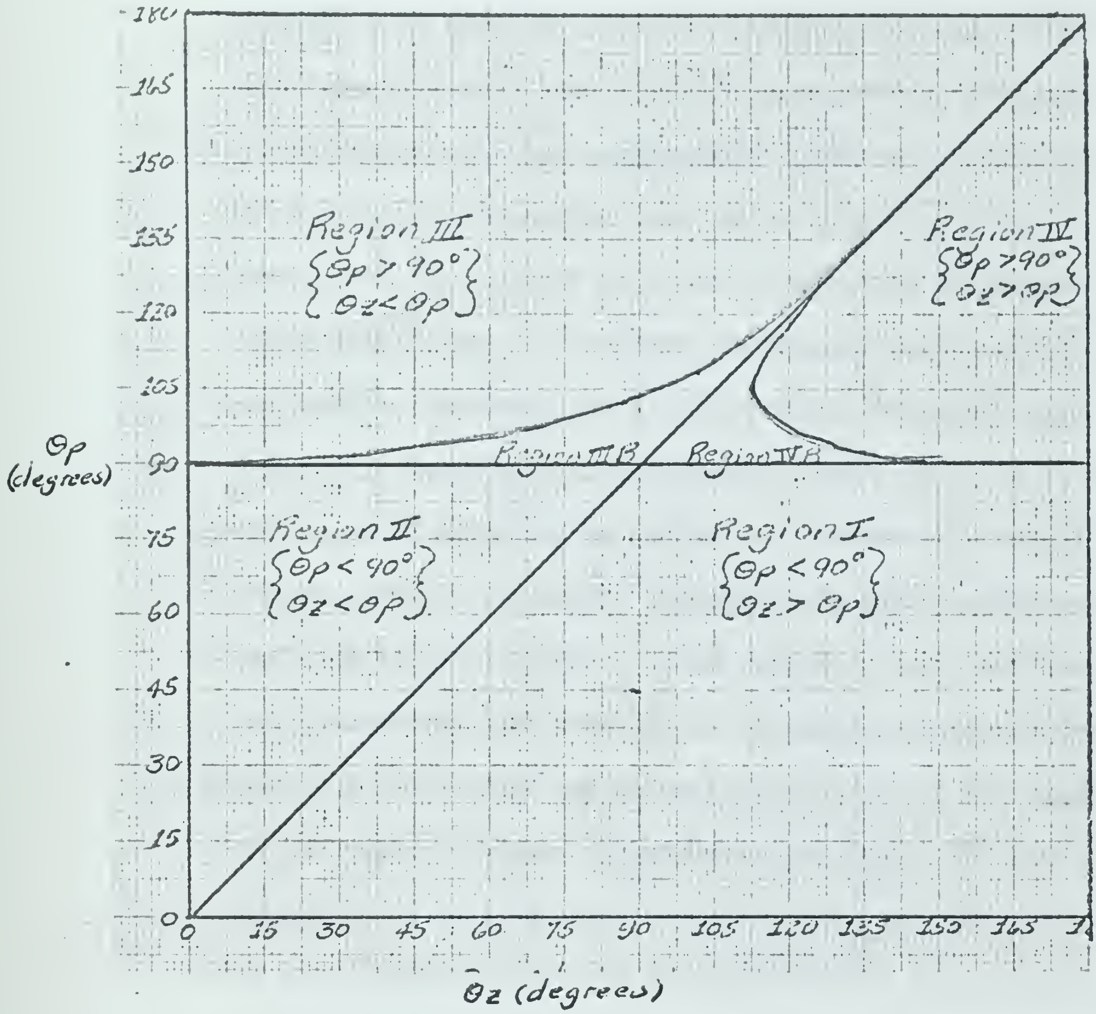


Fig.11: Regions of Interest on θp vs. θz Plot.

at only a finite number of points and a finite number of extremals exist (this finite number may be zero).

(4) $\theta p > \theta z$ implies that the value $T+\theta z-\theta p$ lies to the left of the value T on the computational diagram. If this condition is coupled with the condition that the slope of the computational line is positive then the only manner in which the equality expressed by Equation II can exist is for the computational line to "pierce" the curve of Sin T (the Sin T curve is concave downward; therefore, "piercing" means to intersect on the positive slope side and go "inside" the curve.) Any line which "pierces" must "emerge"; therefore, the extremals always occur in pairs. If the first extremal is called an "overshoot", then it can be stated that every "overshoot" is accompanied by an "undershoot"; however, it is possible that the time of the "overshoot" and time of the "undershoot" may coincide.

(5) $\theta p < \theta z$ implies that the value $T+\theta z-\theta p$ lies to the right of the value T on the computational diagram. If this condition is coupled with the condition that the slope of the computational line is positive then the equality expressed by Equation III can exist for a "piercing" condition, or it can exist for the condition in which the computational line approaches just to the horizontal distance $Op-Oz$ from the curve Sin T and no closer. If the latter condition occurs then an "overshoot" and an "undershoot"

occur at the same time, defined by the particular value of T, and no extremals will occur after that time; the slope of the time response goes to zero, but does not change sign. The first extremal always exists, but extremals after that time occur in pairs.

These factors are of the utmost importance, and if they are not thoroughly understood the reader should review the procedures outlined in Section 5 for computing Tp. These factors are graphically illustrated in Figures 12, and 13. Figure 12 is a sample computational diagram for $\theta p<\theta z$. The values of Sin T for positive half cycles are drawn. One sample computational line has been constructed for the case $\theta p<90^\circ$ ($\theta p=73^\circ$ for the example). The points where Equation II is satisfied are depicted by short vertical lines on the computational line. Four sample computational lines have been drawn for the case $\theta p>90^\circ$ ($\theta p=108^\circ$ for the example). Each line has been drawn for the same θp but a different value of $\theta z-\theta p$. For line "A" the extremals exist as overshoot-undershoot-overshoot and no further extremals. For line "B" the first overshoot occurs, but the times of the first undershoot and the second overshoot coincide, and no further extremals exist. For line "C" only the first overshoot exists with no further extremals. For line "D" the value of $\theta z-\theta p$ has reached its maximum of 72° (θz maximum is 180° and θp is 108°), and then the first overshoot occurs at Tp=0; no further extremals exist. Figure 13 is a sample computational diagram for $\theta p>\theta z$. The values of Sin T for negative half cycles are drawn. One sample computational

line has been constructed for the case $\theta p<90^{o}$ ($\theta p=73^{o}$ for the example). Four sample computational lines have been drawn for the case $\theta p>90^{o}$ ($\theta p=108^{o}$ for the example). For line "A" two overshoots and two undershoots exist. For line "B" one overshoot and undershoot exist. For line "C" the times of overshoot and undershoot coincide. For line "D" no extremals exist. The properties of the regions on Figure 13 may now be stated in terms of these factors:

(A) Region I, Referencing (1), (2), (3), Line E-Figure 12: All extremals exist, slope of the computational line is negative, and positive half cycles are plotted on the computational diagram.

(B) Region II, Referencing (1), (2), (3), Line E-Figure 13: All extremals exist, slope of the computational line is negative, and negative half cycles are plotted on the computational diagram.

(C) Region III, Referencing (1), (2), (3), (4), Lines A,B,C,D - Figure 13: A finite even number of extremals exist, slope of the computational line is positive, negative half cycles are plotted on the computational diagram. In region IIIA, the computational line will always appear as line D, i.e., there can be no overshoot in this region.

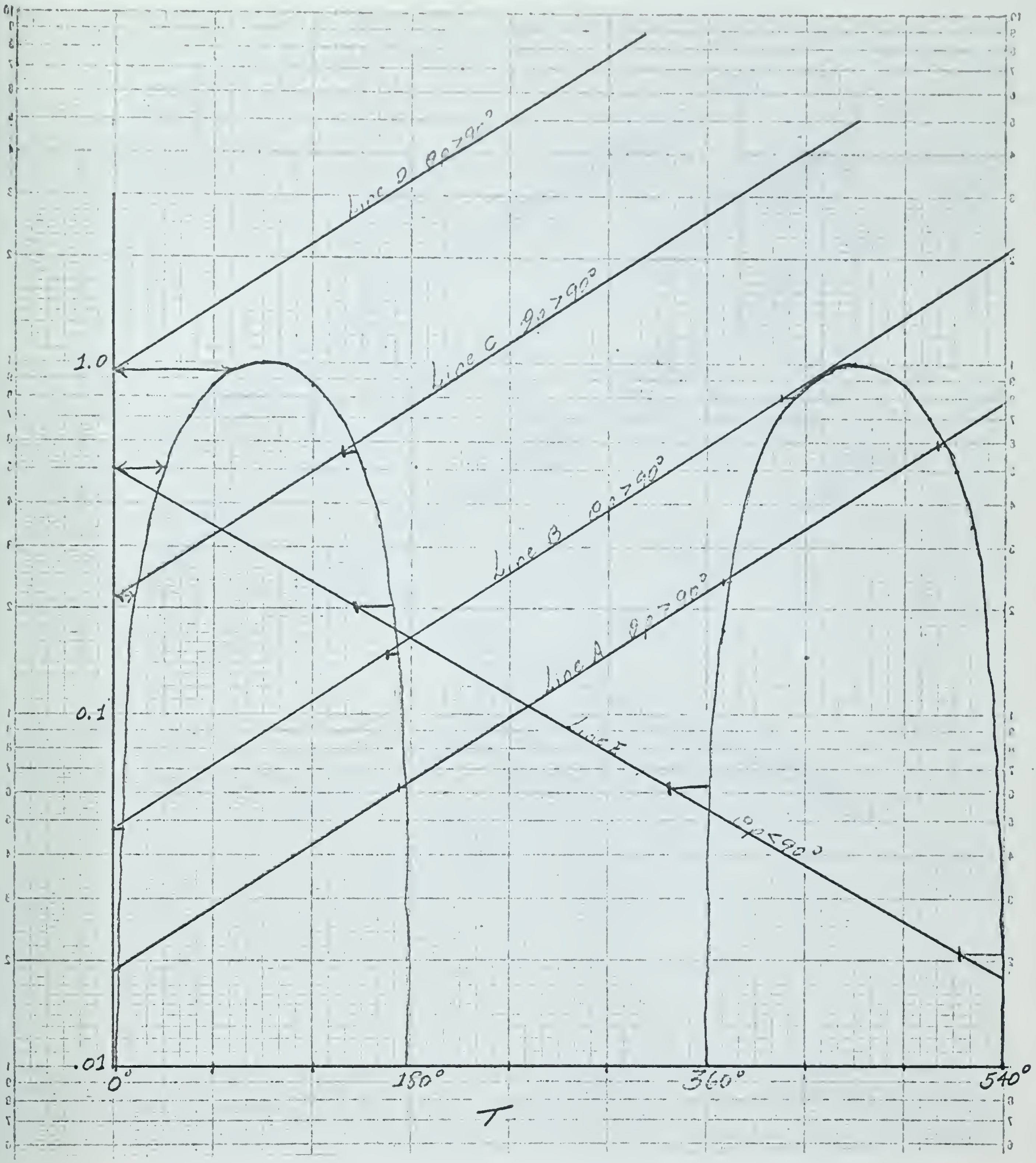


Fig.12: Sample Computational

Diagram($\Theta p<\Theta z$).

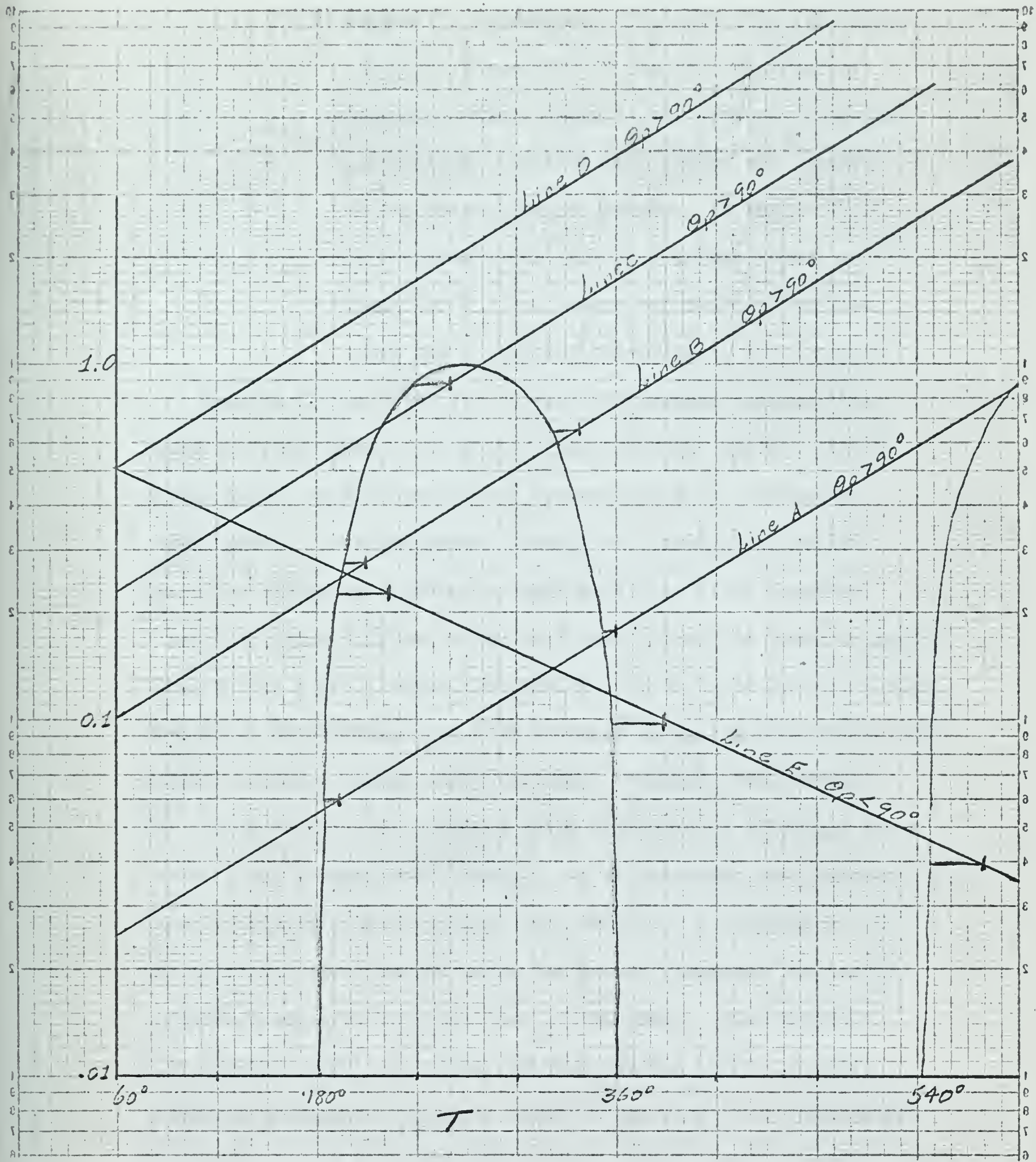


Fig.13: Sample Computational Diagram($\theta p > \theta z$).

(D) Region IV, Referencing (1), (2), (3), (4), Lines A,B,C,D - Figure 12: A finite odd number of extremals exist, slope of the computational line is positive, positive half cycles are plotted on the computational diagram. In region IVA, the computational line will always appear as line C or D, i.e., there can be only one overshoot and no further extremals in this region.

Regions IIIB and IVB are of special interest because they bound the line $\theta z=\theta p$. Along this line the pole and zero of the system cancel each other and the system acts as a pure second order system. An unsuccessful attempt at cancellation would cause the system to operate in regions IIIB or IVB. Consider first the region IIIB and refer to Figure 13 and the computational lines for $\theta p > 90^{\circ}$. Assume that $\theta p=108^{\circ}$ and θz is to be varied from $\theta z=108^{\circ}$ to $\theta z=0^{\circ}$ which will be a movement along the horizontal line $\theta p=108^{\circ}$ through regions IIIB and IIIA. Obviously when $\theta z=\theta p$, $\sin(\theta z-\theta p)=0$ and all extremals exist and occur at multiples of π (this is the second order system). As θz decreases the computational line translates upward rapidly; when $\theta z=106.3^{\circ}$ $[\sin(\theta z-\theta p)= -.03]$ the point has been reached where the second overshoot and second undershoot occur at the same time (Tp=10.6rad). When $\theta z=96.5^{\circ}$ $[\sin(0z-0p)= -.2]$ the point has been reached where the first overshoot and first undershoot occur at the same time (Tp=4.5rad); this point is the dividing line between region IIIB and region IIIA.

For $\theta z < 96.5^\circ$ no overshoot can exist. Consider next the region IVB and refer to Figure 12 and the computational lines for $\theta p > 90^\circ$. Again assume $\theta p = 108^\circ$ and θz is to be varied from $\theta z = 108^\circ$ to $\theta z = 180^\circ$ which will be a movement along the horizontal line $\theta p = 108^\circ$ through regions IVB and IVA. As θz increases the computational line translates upward; when $\theta z = 113.7^\circ$ [Sin $(\theta z - \theta p) = 0.1$] the point has been reached where the first undershoot and second overshoot occur at the same time(Tp =7.35 rad); this point is the dividing line between regions IVB and IVA. For $\theta z > 113.7^\circ$ only one overshoot can occur. When $\theta z = 180^\circ$ [Sin $(\theta z - \theta p) = .95$] The overshoot occurs at its minimum time of Tp=1.57. It is possible to continue the reasoning in this fashion for various values of θp and thus summarize the properties of regions IIIB and IVB. This summary is presented in Table II.

The next factor of interest for the third order system with one zero is the normalized magnitude of the extremals. This information cannot be plotted on a two-dimensional plot since Mpt is a function of θp, θz, and τ . It is not difficult however to predict the order of magnitude of the overshoot. Along the line $\theta p = \theta z$ the system is second order and the normalized magnitude of the overshoot can be read from the second order curve. Along the axis $\theta p = 0^\circ$ the system operates as second order with one zero and the normalized magnitude of the overshoot can be read from Figure 6. Along the axis $\theta z = 0^\circ$ the system operates as pure third order and the normalized magnitude of overshoot can be read from

Region	Defining Conditions	Half-cycles of Sin τ	Slope of the Computational Line	Number of Extremals	Range of T_{p1}	Range of T_{p2}	Range of T_{p3}
I	$\left\{\begin{matrix}\theta_p < 90^\circ \\ \theta_z > \theta_p\end{matrix}\right\}$	Positive	Negative	Infinite	π to 0	2π to π	3π to 2π
II	$\left\{\begin{matrix}\theta_p < 90^\circ \\ \theta_z < \theta_p\end{matrix}\right\}$	Negative	Negative	Infinite	π to 2π	2π to $5\pi/2$	3π to 4π
III	$\left\{\begin{matrix}\theta_p > 90^\circ \\ \theta_z < \theta_p\end{matrix}\right\}$	Negative	Positive	Finite (even)			
IIIA	Same	Same	Same	Zero	(No extremals exist)		
IIIB	Same	Same	Same	Finite (even)	π to $2\pi-$	2π to $\pi+$	3π to 4π
IV	$\left\{\begin{matrix}\theta_p > 90^\circ \\ \theta_z > \theta_p\end{matrix}\right\}$	Positive	Positive	Finite (odd)			
IVA	Same	Same	Same	One	$\pi-$ to 0	(no 2nd or 3rd extremals)	
IVB	Same	Same	Same	Finite (odd)	π to $0+$	2π to $7\pi/3-$	3π to $2\pi+$

Note #1: Range of values of T_{pn} given for $|\theta z-\theta p|$ increasing from zero.

Note #2: Where limit of a range is indicated as $\chi\pm$ the amount indicated by the $\pm$ is dependent on the value of θp. This amount is small for θp close to 90° and large for $\theta p >> 90^\circ$.

Note #3: The constant T_{pn} lines are discontinuous where they intersect the bounderies between regions IIIA and IIIB, IIA and IVB.

TABLE II

Figure 8(a). It is also possible to plot the line along which Mpt is unity. This is accomplished by returning to Eq'n IIIa and equating to final value:

$$M_{pt} = 1 = 1 + \frac{\sin(\theta_p + \tau)}{\sin\tau \, \sin(\theta_z + \tau)} e^{-T_p/\tan\tau} \left[\frac{\sin\theta_p}{\sin(\theta_p + \tau)} \sin(T_p + \theta_z - \theta_p) - \sin(T_p + \theta_z - \theta_p + \tau) \right]$$

$$\therefore \frac{\sin\theta_p}{\sin(\theta_p + \tau)} \sin(T_p + \theta_z - \theta_p) = \sin(T_p + \theta_z - \theta_p + \tau)$$

$$\sin\theta_p \sin(T_p + \theta_z - \theta_p) = \sin(\theta_p + \tau) \sin(T_p + \theta_z - \theta_p + \tau)$$

$$\cos(T_p + \theta_z) = \cos(T_p + \theta_z + 2\tau)$$

Which is satisfied when:

$$T_p = -(\theta_z + \tau) \pm 2n\pi$$

This curve can be plotted by selecting a particular τ and plotting points on the T_p curves by the equation:

$$\theta_z = 2n\pi - \tau - T_p$$

Curves have been plotted on Figures 10a and 10c, for selected values of τ, depicting the curve along which Mpt=1 (first overshoot in Figure 10a and second overshoot in Figure 10c). Note that these curves end at $\theta z = \theta p = 180^\circ - \tau$ which represents the origin of the S-plane. Beyond this point the pole and zero of the system fall in the right half plane, and the system is unstable. A limiting situation in determining magnitude of the extremals is encountered by letting $\theta z = 180 - \tau$ (zero at the origin). When this occurs Sin $(\theta z + \tau) = \text{Sin } 180^\circ = 0$ and the expression for Mpt becomes infinite, but the final value $\left\{\frac{kz}{\omega_n^2 p}\right\}$ goes to zero so that the expression is meaningless. To evaluate the overshoot the value Z is brought inside the parentheses in Equation III:

$$X_{(tp)}=\frac{k}{\omega_n^2 P}\left\{Z+\frac{Z\sin(\theta p+\gamma)}{\sin\gamma\sin(\theta z+\gamma)}e^{-\omega d\,tp/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\omega d\,tp+\theta z-\theta p)-\sin(\omega d\,tp+\theta z-\theta p+\gamma)\right]\right\}$$

$$=\frac{k}{\omega_n^2 P}\left\{Z+\frac{\sin(\theta p+\gamma)}{\sin\gamma}\frac{\omega_n}{\sin\theta z}e^{-\omega d\,tp/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\omega d\,tp+\theta z-\theta p)-\sin(\omega d\,tp+\theta z-\theta p+\gamma)\right]\right\}$$

For the zero at the origin:

$$Z=0\;;\;\theta z=180-\gamma\;;\;\sin\theta z=\sin\gamma$$

$$\therefore X_{(tp)}=\frac{k}{\omega_n^2 P}\left\{\frac{\sin(\theta p+\gamma)}{\sin^2\gamma}\omega_n e^{-\omega d\,tp/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\omega d\,tp+180-\gamma-\theta p)-\sin(\omega d\,tp+180-\theta p)\right]\right\}$$

$$=\frac{-k}{\omega_n P}\left\{\frac{\sin(\theta p+\gamma)}{\sin^2\gamma}e^{-\omega d\,tp/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\omega d\,tp-\theta p-\gamma)-\sin(\omega d\,tp-\theta p)\right]\right\}$$

This equation is good along the line $\theta z=180^{\circ}-\gamma$ until the point $\theta p=180^{\circ}-\gamma$ is reached. Bringing P inside the parentheses:

$$X_{(tp)}=\frac{-k}{\omega_n}e^{-\omega d\,tp/\tan\gamma}\left\{\frac{\sin\theta p}{P\sin^2\gamma}\sin(\omega d\,tp-\theta p-\gamma)-\frac{\sin(\theta p+\gamma)}{P\sin^2\gamma}\sin(\omega d\,tp-\theta p)\right\}$$

$$=\frac{-k}{\omega_n^2}e^{-\omega d\,tp/\tan\gamma}\left\{\frac{\sin^2\theta p}{\sin^2\gamma}\frac{\sin(\omega d\,tp-\theta p-\gamma)}{\sin(\theta p+\gamma)}-\frac{\sin\theta p}{\sin^2\gamma}\sin(\omega d\,tp-\theta p)\right\}$$

But for $\theta z=180^{\circ}-\gamma$, Eq'n. II becomes:

$$\sin(180^{\circ}-\gamma-\theta p)e^{-\omega d\,tp/\tan\theta p}=\sin(\omega d\,tp+180-\gamma-\theta p)$$

$$e^{-\omega d\,tp/\tan\theta p}=-\frac{\sin(\omega d\,tp-\theta p-\gamma)}{\sin(\theta p+\gamma}$$

$$\therefore X_{(tp)}=\frac{-k}{\omega_n^2}e^{-\omega d\,tp/\tan\gamma}\left[-\left(\frac{\sin\theta p}{\sin\gamma}\right)^2 e^{-\omega d\,tp/\tan\theta p}-\frac{\sin\theta p}{\sin^2\gamma}\sin(\omega d\,tp-\theta p)\right]$$

Letting $\theta p=180^{\circ}-\gamma$

$$=\frac{-k}{\omega_n^2}e^{-\omega d\,tp/\tan\gamma}\left[-e^{\omega d\,tp/\tan\theta p}-\frac{\sin(\omega d\,tp+\gamma-180^{\circ})}{\sin\gamma}\right]$$

$$=\frac{k}{\omega_n^2}\left[1+e^{-\pi/\tan\gamma}\right]\qquad(\text{since }\omega d\,tp=\pi)$$

which is the expression obtained for a pure second order system.

Another limiting expression occurs when $\theta p=180^\circ-\gamma$ (pole at the origin). When this occurs $\sin(\theta p+\gamma)=\sin 180^\circ=0$, but the final value $\left\{\frac{kZ}{\omega_n^2 P}\right\}$ becomes infinite so that the expression is meaningless. To evaluate Mpt the value P is brought inside the parentheses as before:

$$X(tp)=\frac{kZ}{\omega_n^2}\left\{\frac{1}{P}+\frac{\sin(\theta p+\gamma)}{P\sin\gamma\sin(\theta z+\gamma)}e^{-\omega d tp/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\omega d tp+\theta z-\theta p)+\sin(\omega d tp+\theta z-\theta p+\gamma)\right]\right\}$$

but $\frac{1}{P}=\frac{\sin\theta p}{\omega_n\sin(\theta p+\gamma)}$

$$X(tp)=\frac{kZ}{\omega_n^3}\left\{\frac{\sin\theta p}{\sin(\theta p+\gamma)}+\frac{\sin\theta p}{\sin\gamma\sin(\theta z+\gamma)}e^{-\omega d tp/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\omega d tp+\theta z-\theta p)+\sin(\omega d tp+\theta z-\theta p+\gamma)\right]\right\}$$

Substituting from Eq'n. II:

$$=\frac{kZ}{\omega_n^3}\left\{\frac{\sin\theta p}{\sin(\theta p+\gamma)}+\frac{\sin\theta p}{\sin\gamma\sin(\theta z+\gamma)}e^{-\omega d tp/\tan\gamma}\left[\frac{\sin\theta p}{\sin(\theta p+\gamma)}\sin(\theta z-\theta p)e^{-\omega d tp/\tan\theta p}+\sin(\omega d tp+\theta z-\theta p+\gamma)\right]\right\}$$

Letting $\theta p=180-\gamma$; $\sin\theta p=\sin\gamma$; $\tan\theta p=-\tan\gamma$:

$$X(tp)=\frac{kZ}{\omega_n^3}e^{-\omega d tp/\tan\gamma}\frac{\sin(\omega d tp+\theta z+2\gamma)}{\sin(\theta z+\gamma)}$$

This equation is good along the line $\theta p=180^\circ-\gamma$ until the point $\theta z=180^\circ-\gamma$ is reached. Bringing Z inside the parentheses and recognizing that: $Z=\frac{\omega_n\sin(\theta z+\gamma)}{\sin\theta z}$

$$X(tp)=\frac{k}{\omega_n^2}e^{-\omega d tp/\tan\gamma}\frac{\sin(\omega d tp+\theta z+2\gamma)}{\sin\theta z}$$

Letting $\theta z=180-\gamma$; $\sin\theta z=\sin\gamma$:

$$X(tp)=k/\omega_n^2\, e^{-\omega d tp/\tan\gamma}$$

But $\omega d tp=\pi$:

$$\therefore X(tp)=k/\omega_n^2\, e^{-\pi/\tan\gamma}$$

It may be noted that this is the expression for magnitude of the first overshoot of a second order system with the exception of the additive constant k/ω_n^2.

The character of the variation of the magnitude of the first peak may be summarized by means of a diagram as depicted in Figure 14.

The coordinates are the θz and θp axes used for the τ_p plots. The line $\theta z=\theta p$ has been drawn in. A value of $\tau=60^\circ$ ($\cos\tau=.5$) has been selected arbitrarily as an example. The value of Mpt=1.16 has been determined from the second order equations for placement on the $\theta z=\theta p$ line. From Figure 6 values of Mpt for various θz have been placed along the θz axis. From Figure 8(a) values of Mpt for various θp have been placed along the θp axis. The curve for Mpt=1.0 has been copied from Figure 10a (all values of Mpt assuming final value =1). The curves outlining the regions of no overshoot and of one overshoot only have also been drawn in. The lines $\theta z=180^\circ-\tau$ and $\theta p=180^\circ-\tau$ have been drawn in to depict the right half plane (RHP) and left half plane (LHP). The values of overshoot along these latter lines are determined by the equations derived above.

Approximate curves for Mpt=1.1 and Mpt=1.3 have been sketched in to show the form of the curves. This form is that the curves are convex toward the $\theta z=\theta p$ line and all curves intersect at the origin. The value of Mpt becomes unbounded for values close to the $\theta z=180-\tau$ line and approaches zero along the $\theta p=180-\tau$ line. Mpt is positive in regions I and IV and negative in regions II and III. It is a simple matter to sketch a diagram of this sort for any value of τ and thereby predict the action of the system both in time and magnitude. Similar diagrams for the first undershoot and second overshoot may also be quickly sketched.

No curves have been plotted for rise time (tc) since the computation is straightforward and the information already plotted gives a complete picture of the response of the system.

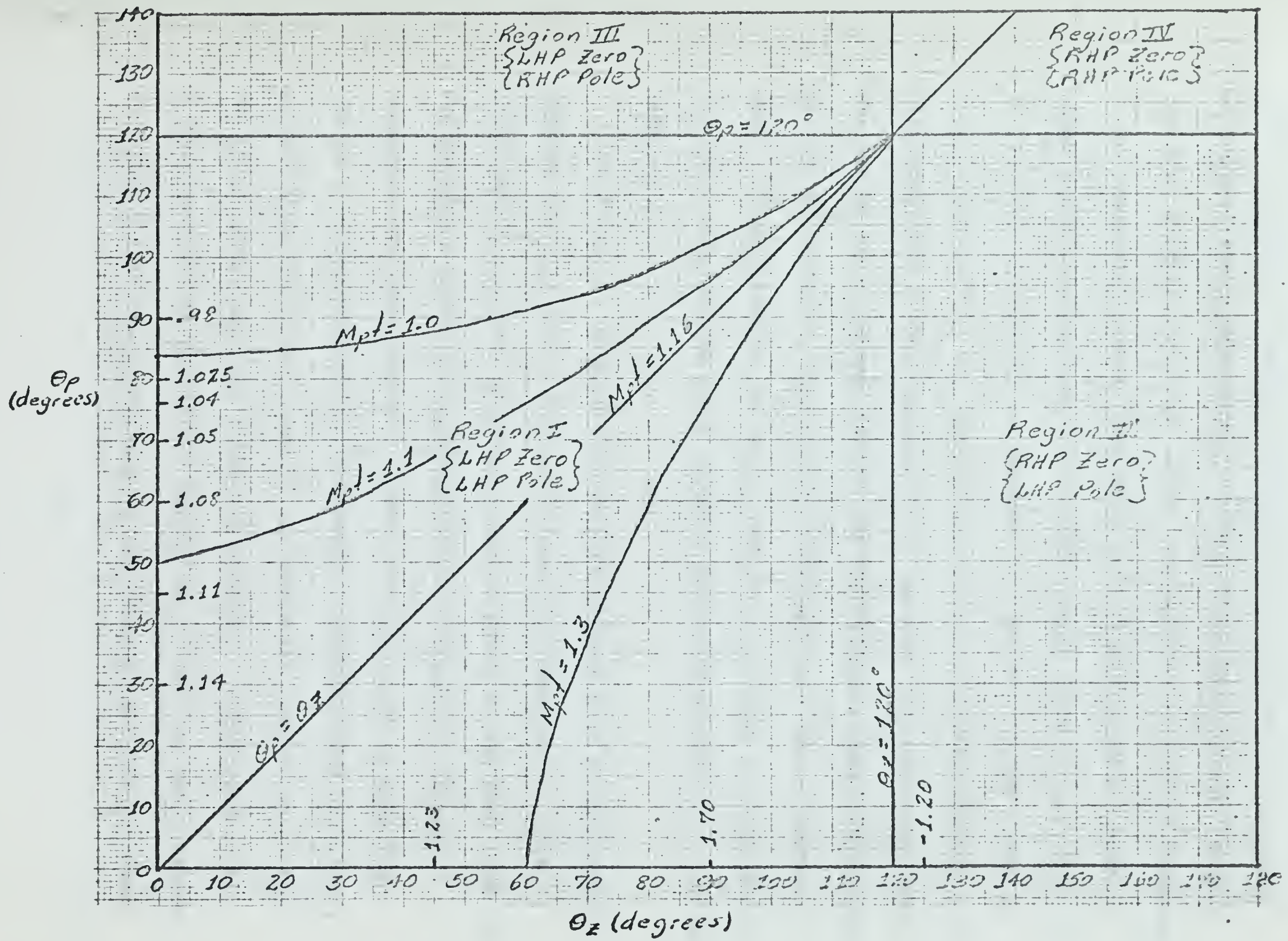


Fig.14: Magnitude of First Extremal for a System with $\gamma = 60°$ (Final value normalized to unity).

10. A New Look at Some Old Concepts: Dominance and Cancellation.

It has been stated repeatedly in this thesis that the most common method of treating the higher order systems is by making a second order approximation. This is sometimes done by forcing the real singularities far out on the negative real axis (dominance) or by cancelling with a pole and a zero (cancellation). Each of these methods is performed in a rather vague manner; i.e., how far is "far out" in the LHP in the case of dominance or how close is "close together" in the case of cancellation. Another question is: what parameter of the response is of the most importance in making the second order approximation? Is the designer/analyst interested in time of response, magnitude of fluctuations, time of extremals, etc.? It is possible, by making use of the methods of this thesis, to take a new look at these concepts and perhaps make them a little less vague.

Consider the case of a third order system with a troublesome pole on the real axis. Using the concept of dominance a rule of thumb says to make the pole ten times as great as the real part of the conjugate roots, but an examination of Figures 7 through 9 discloses that making $\theta p=30^{o}$ causes a variation of no more than 10% in the magnitudes of the extremals (the magnitudes are less), no more than 15% in the times of the extremals, and no more than 30% in the rise time (the times are greater). While some or all of these variations may be too large, two points are demonstrated:

(1) If the time parameters of the response are the important

factors then θp must be made smaller than if the magnitude parameters are the important factors.

(2) The required distance of the pole from the origin varies with the value of γ. For example, in the above case, if γ=30° then it would only be necessary to make the pole twice as far out as the real part of the conjugate roots; however if γ =60° the pole must be four times as far out as the real part of the conjugate roots.

It is fairly certain that the rule of thumb value of "ten times as great" will make the pole insignificant, but it is equally certain that it is not necessary to be so drastic in all cases. Utilizing the methods described here and armed with the requirements of the situation at hand, the designer can develop the most efficient configuration.

Consider now the same system as above, but assume that the troublesome pole is to be removed by cancellation with a zero. Examination of Figures 10 and 14 show immediately that the requirement of how close the zero must be to the pole to effect cancellation is dependent upon the location of the pole with respect to the conjugate roots, value of the angle γ, and the relative importance of time and magnitude. For $\theta p < 70°$ θz can be as much as 15° either side of the value of θp and the time of extremals will change no more than 10%, and if $\gamma < 70°$ the change in magnitudes will be on the order of 10%. On the other hand, if $\theta p > 90°$ the value of θz becomes critical. This is best illustrated

by example:

Given: $\gamma=60^{\circ}$; $\theta p=110^{\circ}$ (See Figure 16)

θz	Tp	Mpt
115°	2.8	1.958
112°	3.0	1.353
110°	3.14	1.160 (exact cancellation)
108°	3.3	1.037
105°	3.6	.908
100°	4.55	.819
100°	(no extremal)	

For a value of P=.3 this range of θz represents a variation of Z from .199 to .489. It is readily apparent that a very slight error in cancellation can cause a serious deviation from second order operation. It may also be noted an error of $\theta z > \theta p$ causes a smaller change in Tp and a greater change in Mpt than an error of $\theta z < \theta p$. Again, it becomes easy to determine the margin of error allowed to effect the desired response. A quick glance at the charts will determine for the designer his tolerance and which direction to weight the uncertainty.

11. Conclusion.

In order to simplify the analysis of the third order equation with one zero the time response equation has been developed in terms of the angles at the singularities:

$$X_{(t)} = \frac{KZ}{\omega_n^2 P}\left\{1 + \frac{\sin\theta_p \sin(\theta_z - \theta_p)}{\sin\gamma \sin(\theta_z + \gamma)} e^{-\omega_d t/\tan\theta_p} - \frac{\sin(\theta_p+\gamma)}{\sin\gamma \sin(\theta_z+\gamma)} e^{-\omega_d t/\tan\gamma} \sin(\omega_d t + \theta_z - \theta_p + \gamma)\right\}$$

This equation has been manipulated to provide expressions for three time response parameters.

(1) Times of extremals (t_{pn}):

$$\sin(\theta_z - \theta_p) e^{-\omega_d t_{pn}/\tan\theta_p} = \sin(\omega_d t_{pn} + \theta_z - \theta_p)$$

(2) Magnitudes of extremals ($X_{(t_p)}$):

$$X_{(t_p)} = \frac{KZ}{\omega_n^2 P}\left\{1 + \frac{\sin(\theta_p+\gamma)}{\sin\gamma \sin(\theta_z+\gamma)} e^{-\omega_d t_p/\tan\gamma}\left[\frac{\sin\theta_p}{\sin(\theta_p+\gamma)}\sin(\omega_d t_p + \theta_z - \theta_p) - \sin(\omega_d t_p + \theta_z - \theta_p + \gamma)\right]\right\}$$

(3) Times when response crosses the final value (t_{cn}):

$$\frac{\sin\theta_p}{\sin(\theta_p+\gamma)}\sin(\theta_z - \theta_p) e^{-\omega_d t_{cn}/\tan\theta_p} = \sin(\omega_d t_{cn} + \theta_z - \theta_p + \gamma)$$

A simple graphical method has been developed and demonstrated for solving the transcendental equations above.

The equations above have been utilized to analyze four types of systems:

(1) Pure second order

(2) Second order with one zero

(3) Pure third order

(4) Third order with one zero

Examples have been worked and a variety of curves developed to enable the designer and/or analyst to make a rapid appraisal of the response of the various systems.

An analysis has been presented on new methods for achieving dominance or cancellation and evaluating the effects of errors in such attempts.

It is felt that a new insight has been gained into the actions of the third order system with one zero. With the methods and charts of this thesis it is possible to predict immediately the form of the response, and only a short time is required to compute the parameters and sketch the response accurately. For the designer interested in improving the response of a second order system by adding real singularities, the means is now available for doing so exactly without resorting to the trial and error methods.

There is a possibility that the methods of this thesis might be extended to other studies. Some that have occured to the author are listed here in the hope that others may be persuaded to attempt to extend the method:

(1) A generalized method for expressing the time response of higher order systems in terms of the angles at the singularities.

(2) Combination of the method with that of Mitrovic to develop generalized curves for the time response on the Mitrovic Plane.

(3) Extension of the equations of this thesis to develop charts for frequency response parameters.

(4) Combination with methods such as those of Ross-Warren and Pollack to develop improvements and/or simplifications in the methods of compensation calculations.

13. Bibliography.

1. Abbott, William B and Walter B. Patton. Rapid Approximation of Servomehanism Transient Response. MS Thesis, Mass.Inst. of Tech., June, 1961.

2. Thaler, G. J. and R. G. Brown. Analysis and Design of Feedback Control Systems. McGraw-Hill Book Company, Inc., 1960.